Classical and the Bayesian Estimation of Process Capability Index $\mathcal{C}_{py}$: A Comparative Study

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Abstract

In this study, to estimate the process capability index $\mathcal{C}_{py}$ when the process follows different distributions (Lindley, Xgamma, and Akash distributions), I have used five methods of estimation, namely, the maximum likelihood method of estimation, the least and weighted least squares method of estimation, the maximum product of spacings method of estimation, and the Bayesian method of estimation. The Bayesian estimation is studied for symmetric loss function with the help of the Metropolis-Hastings algorithm method. The Metropolis-Hastings algorithm approach is used to study Bayesian estimation for symmetric loss functions. Four bootstrap approaches and Bayesian methods are used to create confidence intervals for the index $\mathcal{C}_{py}$. Based on their respective MSEs/risks for point estimates of $\mathcal{C}_{py}$ and average widths ($\mathcal{AW}s$) for interval estimates, I have investigated the performance of various estimators. To assess the accuracy of the various approaches, Monte Carlo simulations are conducted. It is found that the Bayes estimates performed better than the considered classical estimates in terms of their

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corresponding risks. To illustrate the performance of the proposed methods, two real data sets are analyzed.

Keywords: Bootstrap confidence interval, process capability index, Lindley distribution, Xgamma distribution, Akash distribution.

1 Introduction

Effective management and evaluation of output service quality is a prominent topic in the manufacturing industry. The most generally used indices to judge the processes appear to be process capability indices (PCIs), which are particularly popular among industries for evaluating (manufacturing) processes since they are dimensionless, easy to read, and comprehensible. Despite their flaws, these indexes are frequently employed in a range of industries, owing to the single-number summary's simplicity and attraction to engineers and management. The most commonly utilised PCIs are $\mathcal{C}_p$,, $\mathcal{C}_{pk}$, $\mathcal{C}_{pmk}$, and Cpm [see Juran (1974), Kane (1986), Chan et al. (1988), and Pearn et al. (1992)]. They are predicated on the assumption that a given process may be characterised by a normal probability model with a process mean μ and standard deviation σ. Furthermore, in so many industrial and service activities, the assumption of normalisation is basically a simplifying notion that is frequently inaccurate [see, Gunter (1989)]. In their recent work, Maiti et al. (2010) obtained a generalized process capability index (GPCI) Cpy in their recent work. The index's attractiveness is that it is closely linked to the vast majority of PCIs defined in the literature. Furthermore, it includes both normal and non-normal random variables, as well as continuous and discrete random variables, and is described as follows:

$$\begin{aligned}\mathcal{C}_{py} &= \frac{F(U) - F(L)}{F(UDL) - F(LDL)} \\ &= \frac{p}{p_0},\end{aligned}$$

where $F(t) = P(Z \leq t)$ is the cumulative distribution function of the quality characteristic Z. The lower and upper specification limits are L and U, respectively, whereas p is the process yield and p_0 is the ideal yield. LDL and UDL are the lower and higher acceptable thresholds, respectively.

To draw the inference about PCIs, quality control engineers generally use the point and the interval estimation. The point estimator is employed to the process performance but in the case of variability in estimation, researchers also on confidence interval (CI) (see, Chan et al. 1988, Smithson (2001)). There are several techniques available in the literature to construct CIs like the bootstrap technique. This technique is a re-sampling method and free from distributional assumptions. Firstly, Efron (1979) introduced this technique. Franklin and Wasserman (1991) employed this technique for the construction of CIs of the PCI $\mathcal{C}_{pk}$. Tong and Chen (1998) likewise utilized bootstrap simulation methods to calculate lower confidence limits for the said indices $\mathcal{C}_p$, $\mathcal{C}_{pk}$ and $\mathcal{C}_{pm}$ when the process distributions were non-normal. Many researchers have already used this approach for other PCIs [see, for reference, Pearn et al. (2014, 2016); Rao et al. (2016); Dey et al. (2021); Saha et al. (2018, 2020a, 2020b); Kumar (2021)].

PCIs are analyzed and studied from both the Bayesian and classical perspectives. Nevertheless, many statisticians prefer the use of the Bayesian approach over the classical approach. When the actual distribution is normally distributed, Saxena and Singh (2006) address the Bayesian estimation of the PCI $\mathcal{C}_p$. Credible intervals for several PCIs were determined by Ouyang et al. (2002) and Lin et al. (2011). One can find the advantages and justification of the Bayesian approach in the works of Chan et al. (1988), Cheng and Spiring (1989), and Shiau et al. $(1999a, 1999b)$. Besides, several authors have discussed Bayesian estimation of the PCIs for many lifetime distributions. Readers may refer to the works of Huiming et al. (2007), Miao et al. (2011), Pearn et al. (2015), Seifi and Nezhad (2017), Saha et al. (2019), Leiva et al. (2014), Perakis and Xekalaki (2002) among others.

The following are the three goals of this paper: First, I have estimate $\mathcal{C}_{py}$ using four distinct classical and Bayesian estimation approaches for various models. To estimate the parameter(s) of various distributions, I have selected four traditional estimation methods: maximum likelihood estimation (MLE), least square estimation (LSE), weighted least square estimation (WLSE), and maximum product spacing estimation (MPSE). Performance is not simply measured in terms of mean square error (MSE); another sort of risk is also employed. The second goal is to compute four bootstrap confidence intervals (BCI) of $\mathcal{C}_{py}$ using the traditional techniques of estimation mentioned above: standard bootstrap ($\mathcal{SB}$), percentile bootstrap ($\mathcal{PB}$), student's t bootstrap ($\mathcal{STB}$), and bias-corrected percentile bootstrap ($\mathcal{BCPB}$). The estimated average widths ($\mathcal{AW}s$) of the BCIs are used to highlight their performance. The final goal is to derive Bayes estimates of the PCI $\mathcal{C}_{py}$ under a symmetric function using gamma priors for the model's parameters. The Metropolis-Hastings (M-H) method is used to calculate Bayes estimates.

We then calculate Bayes credible intervals and compare them to the BCIs. To the best of our knowledge, no research has been conducted to investigate the PCIs $\mathcal{C}_{py}$ employing four BCIs based on the aforementioned classical and Bayesian estimation techniques for the considered distributions. The study's goal is to create a guideline for selecting the optimum way of estimating the indices, which I believe would be of great relevance to applied statisticians and quality control engineers in situations where the item/subgroup quality characteristic follows studied distributions.

The following is how the rest of the article is organized: Section 2 defines GPCI $\mathcal{C}_{py}$ for the distributions under consideration. In addition, I have explain various traditional estimation methods (MLE, LSE, WLSE, and MPSE) for the index $\mathcal{C}_{py}$. Section 3 addresses BCIs such as $\mathcal{SB}$, $\mathcal{PB}$, $\mathcal{STB}$ and $\mathcal{BCPB}$ that are based on the aforementioned GPCI $\mathcal{C}_{py}$ assessment procedures. In section 4, I derive Bayesian estimates of the index $\mathcal{C}_{py}$ using the squared error loss function (SELF) and the highest posterior density (HPD) credible interval. In Section 5, a Monte Carlo simulation experiment was undertaken to evaluate the performances of the aforementioned classical and Bayes estimators of the index $\mathcal{C}_{py}$ in terms of their associated MSEs and risks. Section 6 pointed out two real-life data sets for promotional purposes, and Section 7 includes the study's conclusion.

2 Estimation of Generalized Process Capability Index $\mathcal{C}_{py}$

Here, I have derived the MLE, LSE, WLSE, MPSE, and BCIs of GPCI $\mathcal{C}_{py}$ for some finite mixture distributions, viz., the LnD, XgD, and AkD, respectively.

2.1 Lindley Distribution

The LnD [See, Lindley (1958), Ghitany et al. (2008)] belongs to the exponential family and it can be written as a mixture of exponential and gamma distributions. Suppose Y is a random variable (RV) that follows the LnD(ψ). Then, its probability density function (PDF) and cumulative density function (CDF) are, respectively, given as

$$f(y;\psi) = \frac{\psi^2}{\psi+1}(1+y)e^{-\psi y}; y > 0, \psi > 0 \tag{1}$$

$$F(y;\psi) = 1 - \left[1 + \frac{\psi y}{1+\psi}\right] e^{-\psi y}. \tag{2}$$

Now, GPCI $\mathcal{C}_{py}$, where the quality characteristic follows the LnD, is given as

$$\mathcal{C}_{py} = \frac{[(1+\frac{\psi L}{\psi+1})e^{-\psi L}] - [(1+\frac{\psi U}{\psi+1})e^{-\psi U}]}{p_0} \tag{3}$$

Given a random sample (RS) $Y_1, Y_2, \ldots,\ Y_n$ of size n, drawn from the LnD(ψ) given in Equation (1), the corresponding log-likelihood function ($\ell = \log L(\psi; Y)$) is given as

$$\ell = 2n\log\psi - n\log(\psi+1) + \sum_{i=1}^{n}\log(1+y_i) - \psi\sum_{i=1}^{n} y_i \tag{4}$$

By solving the ensuing equation, we will get the MLE of ψ, say, $\hat{\psi}_{mle}$

$$\frac{\partial\ell}{\partial\psi} = \frac{2n}{\psi} - \frac{n}{1+\psi} - \sum_{i=1}^{n} y_i = 0.$$

Thus, MLE of the parameter ψ is given by [see, Ghitany et al. (2008)]

$$\hat{\psi}_{mle} = \frac{-(\bar{y}-1) + \sqrt{(\bar{y}-1)^2 - 8\bar{y}}}{2\bar{y}} \tag{5}$$

The MLE of $\mathcal{C}_{py}$, denoted by $\hat{\mathcal{C}}_{py}^{mle}(LnD)$, can be obtained by operating the invariance property of MLE, which is given as

$$\hat{\mathcal{C}}_{py}^{mle}(LnD) = \frac{(1+\frac{L\hat{\psi}_{mle}}{1+\hat{\psi}_{mle}})e^{-L\hat{\psi}_{mle}} - (1+\frac{U\hat{\psi}_{mle}}{1+\hat{\psi}_{mle}})e^{-U\hat{\psi}_{mle}}}{p_0}. \tag{6}$$

LSE and WLSE

The LSE and WLSE were proposed by Swain et al. (1988) to estimate the parameters of the Beta distribution. Suppose $F(y_{(j:n)})$ denotes the CDF of the ordered random variables $y_{(1:n)} < y_{(2:n)} < \cdots < y_{(n:n)}$, where, $\{y_{1:n}, y_{2:n}, \ldots, y_{n:n}\}$ is a random sample of size n from a distribution function $F(\cdot)$. As a result, the LSEs of (ψ), say, ($\hat{\psi}_{lse}$) can be found by reducing

$$L(\psi) = \sum_{i=1}^{n}\left[F(y;\psi) - \frac{i}{n+1}\right]^2$$

with respect to ψ, where $F(y;\psi)$ is the CDF of the distribution. Equivalently, it can also be obtained by solving the following non-linear equation

$$\sum_{i=1}^{n}\left[1-\left(1+\frac{\psi y}{\psi+1}\right)e^{-\psi y}-\frac{i}{n+1}\right]\Delta_1(y;\psi)=0$$

where $\Delta_1(y;\psi)$ is the first derivative of the respective distribution

$$\Delta_1(y;\psi)=\frac{ye^{-\psi y}}{(\psi+1)^2}[\psi^2(y+1)+\psi(y+2)] \tag{7}$$

Thus, the LSE for GPCIs under LnD can be obtained by replacing ψ with $\hat{\psi}_{lse}$ in Equation (3) and can be given as

$$\hat{C}_{py}^{lse}=\frac{[(1+\frac{L\hat{\psi}_{lse}}{1+\hat{\psi}_{lse}})e^{-L\hat{\psi}_{lse}}]-[(1+\frac{U\hat{\psi}_{lse}}{1+\hat{\psi}_{lse}})e^{-U\hat{\psi}_{lse}}]}{P_0} \tag{8}$$

Therefore, in this case, the WLSE of ψ say $\hat{\psi}_{wlse}$ can be obtained by minimizing

$$W(\psi)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(y;\psi)-\frac{i}{n+1}\right]^2$$

to ψ. The estimators can be obtained by differentiating $W(\psi)$ for ψ, and equating to zero.

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\left(1-\left(1+\frac{\psi y}{\psi+1}\right)e^{-\psi y}-\frac{i}{n+1}\right)\right]^2\Delta_1(y;\psi)=0$$

where, $\Delta_1(y;\psi)$ is given in equation 7. Thus, the Process Capability Indices of the above mentioned distribution for WLSE obtained by replacing ψ by $\hat{\psi}_{wlse}$ in Equation (3).

$$\hat{C}_{py}^{wlse}=\frac{[(1+\frac{L\hat{\psi}_{wlse}}{1+\hat{\psi}_{wlse}})e^{-L\hat{\psi}_{wlse}}]-[(1+\frac{U\hat{\psi}_{wlse}}{1+\hat{\psi}_{wlse}})e^{-U\hat{\psi}_{wlse}}]}{P_0} \tag{9}$$

MPSE

Cheng and Amin (1979) proposed the maximum product spacing method as an alternative to MLE for estimating unknown parameters of continuous univariate distributions. Ranneby (1984) independently developed this method as an approximation to the Kullback-Leibler information measure. Cheng and Amin (1983) demonstrated that this method is equally efficient as the MLE and consistent under more broad settings, which influenced our decision. Let us begin by defining

$$D(\alpha;\lambda) = F(y_{i:n}|\alpha,\lambda) - F(y_{i-1:n}|\alpha,\lambda), \quad i = 1,2,\ldots,n+1 \quad (10)$$

where $F(y_{0:n}|\psi) = 0$ and $F(y_{n+1:n}|\psi) = 1 - F(y_{n:n}|\psi)$. Clearly, $\sum_{i=1}^{n+1} D(\psi) = 1$. The MPSEs of the parameter (α,ψ), say, $(\hat{\alpha}_{mpse}, \hat{\psi}_{mpse})$ are obtained by the maximizing the geomatric mean of the spacings with respect to ψ as

$$GM = \left[\prod_{i=1}^{n+1} D_i(\psi)\right]^{\frac{1}{n+1}}$$

or equivalently, by maximizing the function

$$H = logGM = \frac{1}{n+1}\sum_{i=1}^{n+1} logD_i(\psi)$$

with respect to α and λ. The estimates of ψ is obtained by solving the non-linear equations

$$\frac{\delta H}{\delta\psi} = \frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\psi)}\frac{\delta D_i(\psi)}{\psi} = 0$$

where,

$$\frac{\delta H}{\delta\psi} = \frac{F(y_{i:n})|\psi}{\psi} - \frac{F(y_{i-1:n})|\psi}{\psi}$$

can be obtained.

Thus, the GPCI of the above mentioned distribution for MPSE obtained by replacing ψ by $\hat{\psi}_{mpse}$ in Equation (3).

$$\hat{C}_{py}^{mpse} = \frac{[(1+\frac{L\hat{\psi}_{mpse}}{1+\hat{\psi}_{mpse}})e^{-L\hat{\psi}_{mpse}}] - [(1+\frac{U\hat{\psi}_{mpse}}{1+\hat{\psi}_{mpse}})e^{-U\hat{\psi}_{mpse}}]}{P_0} \quad (11)$$

2.2 Xgamma Distribution

The XgD is a new probability distribution derived from a particular finite mixing of exponential and gamma distributions [see, sen et al. (2016)]. If the

PDF and CDF of a continuous RV Y are of the form, it is said to follow an XgD.

$$f(y;\psi) = \frac{\psi^2}{1+\psi}(1+\frac{\psi}{2}y^2)e^{-\psi y}; y > 0, \psi > 0 \tag{12}$$

$$F(y;\psi) = 1 - \frac{(1+\psi+\psi y+\frac{\psi^2 y^2}{2})e^{-\psi y}}{1+\psi}; y > 0, \psi > 0 \tag{13}$$

Now GPCI $\mathcal{C}_{py}$, where the quality characteristic follows the XgD, is given as

$$\mathcal{C}_{py} = \left\{ \frac{[\frac{(1+\psi+\psi L+\frac{\psi^2 L^2}{2})e^{-\psi L}}{1+\psi}] - [\frac{(1+\psi+\psi U+\frac{\psi^2 U^2}{2})e^{-\psi U}}{1+\psi}]}{p_0} \right\} \tag{14}$$

Given a RS $Y_1, Y_2, \ldots, Y_n$ of size n, drawn from the XgD(ψ) given in Equation (12), the corresponding log-likelihood function is given as

$$\ell = 2n\log\psi - n\log(1+\psi) + \sum_{i=1}^{n}\log(1+\frac{\psi}{2}y_i^2) - \psi\sum_{i=1}^{n} y_i \tag{15}$$

By solving the ensuing equation, we will get the MLE of ψ, say, $\hat{\psi}_{mle}$

$$\frac{2n}{\psi} - \frac{n}{(1+\psi)} + \sum_{i=1}^{n}\frac{\frac{y_i^2}{2}}{(1+\frac{\psi}{2}y_i^2)} = \sum_{i=1}^{n} y_i \tag{16}$$

The MLE $\hat{\psi}_{mle}$ of the unknown parameters ψ can be obtained by optimizing the log-likelihood function concerning the involved parameters. In this regard, one can use the packages like, $nlm()$ and/or $maxLik()$ packages of the R software [see Dennis and Schnabel (1983), Henningsen and Toomet (2010)]. Alternatively, the parameters can be obtained by solving the above non-linear Equation (16) with the help of an iterative procedure like the Quasi Newton-Raphson method. Hence, the MLE of the GPCI $\mathcal{C}_{py}$ is obtained by using the invariance property of MLE, of given as

$$\hat{\mathcal{C}}_{py}^{mle}(XgD) = \left\{ \frac{\left[\frac{(1+\hat{\psi}_{mle}+L\hat{\psi}_{mle}+\frac{L^2\hat{\psi}^2_{mle}}{2})e^{-L\hat{\psi}_{mle}}}{1+\hat{\psi}_{mle}}\right] - \left[\frac{(1+\hat{\psi}_{mle}+U\hat{\psi}_{mle}+\frac{U^2\hat{\psi}^2_{mle}}{2})e^{-U\hat{\psi}_{mle}}}{1+\hat{\psi}_{mle}}\right]}{P_0} \right\} \tag{17}$$

LSE and WLSE

Now using the theory of the LSE and WLSE has given in Subsection 2.1, we can get the expressions for XgD as

$$L(\psi) = \sum_{i=1}^{n} \left[1 - \frac{(1 + \psi + \psi y + \frac{\psi^2 y^2}{2}) e^{-\psi y}}{1 + \psi} - \frac{i}{n+1} \right]^2 \Delta_2(y;\psi) = 0$$

where $\Delta_2(y;\psi)$ is the first derivative of the respective distribution,

$$\Delta_2(y;\psi) = \frac{y e^{-\psi x}}{2(1+\psi)^2} [2(2+\psi) + \psi y(1 + y + \psi y)] \tag{18}$$

Thus, the LSEs of GPCIs for the respective distribution can be obtained by replacing ψ with $\hat{\psi_{lse}}$ in Equation (14).

$$\hat{C}_{py}^{lse} = \left\{ \frac{\left[\frac{(1+\hat{\psi}_{lse} + L\hat{\psi}_{lse} + \frac{L^2 \hat{\psi^2}_{lse}}{2}) e^{-L\hat{\psi}_{lse}}}{1+\hat{\psi}_{lse}} \right] - \left[\frac{(1+\hat{\psi}_{lse} + U\hat{\psi}_{lse} + \frac{U^2 \hat{\psi^2}_{lse}}{2}) e^{-U\hat{\psi}_{lse}}}{1+\hat{\psi}_{lse}} \right]}{P_0} \right\} \tag{19}$$

Similarly, for XgD the WLSE of ψ say $\hat{\psi}_{wlse}$ can be obtained by solving the following expression

$$\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[1 - \frac{(1 + \psi + \psi y + \frac{\psi^2 y^2}{2}) e^{-\psi y}}{1 + \psi} - \frac{i}{n+1} \right]^2 \Delta_2(y;\psi) = 0$$

where, $\Delta_2(y;\psi)$ are given in Equation (18). Thus, the WLSEs of GPCIs for the above-mentioned distribution can be obtained by replacing ψ with $\hat{\psi}_{wlse}$ in Equation (14).

$$\hat{C}_{py}^{wlse} = \left\{ \frac{\left[\frac{(1+\hat{\psi}_{wlse} + L\hat{\psi}_{wlse} + \frac{L^2 \hat{\psi^2}_{wlse}}{2}) e^{-L\hat{\psi}_{wlse}}}{1+\hat{\psi}_{wlse}} \right] - \left[\frac{(1+\hat{\psi}_{wlse} + U\hat{\psi}_{wlse} + \frac{U^2 \hat{\psi^2}_{wlse}}{2}) e^{-U\hat{\psi}_{wlse}}}{1+\hat{\psi}_{wlse}} \right]}{P_0} \right\} \tag{20}$$

MPSE

Similarly, using the theory of the MPSE has given in Subsection 2.1, The MPSEs of GPCIs for XgD can obtain by replacing ψ with $\hat{\psi}_{mpse}$ in Equation (14).

$$\hat{\mathcal{C}}_{py}^{mpse} = \left\{ \frac{\left[\frac{(1+\hat{\psi}_{mpse}+L\hat{\psi}_{mpse}+\frac{L^2\hat{\psi^2}_{mpse}}{2})e^{-L\hat{\psi}_{mpse}}}{1+\hat{\psi}_{mpse}} \right] - \left[\frac{(1+\hat{\psi}_{mpse}+U\hat{\psi}_{mpse}+\frac{U^2\hat{\psi^2}_{mpse}}{2})e^{-U\hat{\psi}_{mpse}}}{1+\hat{\psi}_{mpse}} \right]}{P_0} \right\} \tag{21}$$

2.3 Akash distribution

The AkD [see Shanker (2015)] is a novel probability distribution derived from a particular finite mixing of exponential and gamma distributions. The revised one-parameter lifespan distribution's PDF can be written as follows:

$$f(y;\psi) = \frac{\psi^3}{\psi^2+2}(1+y^2)e^{-\psi y}; y > 0, \psi > 0 \tag{22}$$

and, the corresponding CDF is given by

$$F(y;\psi) = 1 - \left[1 + \frac{\psi y(\psi y+2)}{\psi^2+2}\right] e^{-\psi y}; y > 0, \psi > 0 \tag{23}$$

Now GPCI $\mathcal{C}_{py}$, where the quality characteristic follows the AkD, is given as

$$\mathcal{C}_{py} = \frac{[1+\frac{\psi L(\psi L+2)}{\psi^2+2}]e^{-\psi L} - [1+\frac{\psi U(\psi U+2)}{\psi^2+2}]e^{-\psi U}}{p_0} \tag{24}$$

Given a RS $Y_1, Y_2, \ldots, Y_n$ of size n, drawn from the AkD(ψ) given in Equation (22), the corresponding log-likelihood function is given as

$$\ell = 3n\log\psi - n\log(\psi^2+2) + \sum_{i=1}^{n}\log(1+y_i^2) - \psi\sum_{i=1}^{n} y_i \tag{25}$$

The MLE of ψ, say, $\hat{\psi}_{mle}$ can be obtained as the solution of the following equation

$$\psi^3\bar{y} - \psi^2 + 2\psi\bar{y} - 6 = 0$$

Again to obtain the MLE $\hat{\psi}_{mle}$ of the unknown parameter ψ, one can use the techniques mentioned above. After obtaining the MLE of the parameter ψ, the MLE of $\mathcal{C}_{py}$, denoted by $\hat{\mathcal{C}}_{py}^{mle}(AkD)$ can be obtained by operating the invariance property of MLE and which is given as

$$\hat{\mathcal{C}}_{py}^{mle}(AkD) = \left[\frac{\left[1+\frac{L\hat{\psi}_{mle}(L\hat{\psi}_{mle}+2)}{2+\hat{\psi}^2{}_{mle}}\right]e^{-L\hat{\psi}_{mle}} - \left[1+\frac{U\hat{\psi}_{mle}(U\hat{\psi}_{mle}+2)}{2+\hat{\psi}^2{}_{mle}}\right]e^{-U\hat{\psi}_{mle}}}{P_0}\right] \tag{26}$$

LSE and WLSE

Similarly, using the theory of the LSE and WLSE has given in Subsection 2.1, the LSE and WLSE of AkD can also be obtained by solving the following non-linear equation

$$L(\psi) = \sum_{i=1}^{n}\left[1-(1+\frac{\psi y(\psi y+2)}{\psi^2+2})e^{-\psi y} - \frac{i}{n+1}\right]\Delta_3(y;\psi)$$

where $\Delta_3(y;\psi)$ is the first derivative of the respective distribution,

$$\Delta_3(y;\psi) = \frac{e^{-\psi y}}{(\psi^2+2)}\psi y[\psi^3(1+y^2)+2\psi(\psi y+y^2+3)] \tag{27}$$

Thus, the LSE for GPCIs under AkD can obtain by replacing ψ with $\hat{\psi}_{lse}$ in Equation (24) and be given as

$$\hat{\mathcal{C}}_{py}^{lse} = \frac{[1+\frac{L\hat{\psi}_{lse}(L\hat{\psi}_{lse}+2)}{2+\hat{\psi}^2{}_{lse}}]e^{-L\hat{\psi}_{lse}} - [1+\frac{U\hat{\psi}_{lse}(U\hat{\psi}_{lse}+2)}{2+\hat{\psi}^2{}_{lse}}]e^{-U\hat{\psi}_{lse}}}{P_0} \tag{28}$$

Similarly, for XgD the WLSE of ψ say $\hat{\psi}_{wlse}$ can be obtained by solving the following expression

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)} \left[1-\left[1+\frac{\psi y(\psi y+2)}{\psi^2+2}\right]e^{-\psi y} - \frac{i}{n+1}\right]^2 \Delta_2(y;\psi) = 0$$

where, $\Delta_2(y;\psi)$ s given in Equation (27). Thus, WLSEs of the GPCIs of the AkD can obtain by replacing ψ by $\hat{\psi}_{wlse}$ and can be given as

$$\hat{\mathcal{C}}_{py}^{wlse} = \frac{\left[1 + \frac{L\hat{\psi}_{wlse}(L\hat{\psi}_{wlse}+2)}{2+\hat{\psi}^2{}_{wlse}}\right] e^{-L\hat{\psi}_{wlse}} - \left[1 + \frac{U\hat{\psi}_{wlse}(U\hat{\psi}_{wlse}+2)}{2+\hat{\psi}^2{}_{wlse}}\right] e^{-U\hat{\psi}_{wlse}}}{P_0} \tag{29}$$

MPSE

Similarly from Subsection 2.1, the MPSEs of GPCIs for AkD can be obtained by replacing ψ with $\hat{\psi}_{mpse}$ in Equation (24).

$$\hat{\mathcal{C}}_{py}^{mpse} = \frac{\left[1 + \frac{L\hat{\psi}_{mpse}(L\hat{\psi}_{mpse}+2)}{2+\hat{\psi}^2{}_{mpse}}\right] e^{-L\hat{\psi}_{mpse}} - \left[1 + \frac{U\hat{\psi}_{mpse}(U\hat{\psi}_{mpse}+2)}{2+\hat{\psi}^2{}_{mpse}}\right] e^{-U\hat{\psi}_{mpse}}}{P_0} \tag{30}$$

3 Bootstrap Confidence Interval

Efron created the principle of bootstrap re-sampling approach (1979) [See Efron (1979)]. We can create inferential statistics related to the underlying distribution using a simple re-sampling procedure in this approach. Efron (1982), Hall (2013), and Davison and Hinkley provide in-depth treatments of the theoretical development of the bootstrap approach (1997). BCIs have recently been utilised by numerous researchers to create confidence intervals for various PCIs [see, for example, Chatterjee and Qiu (2009); Li et al. (2016), Rao et al. (2016), Kumar et al. (2019; 2021), Kumar and Saha (2020)].

Here, I have obtained four BCIs, namely, $\mathcal{SB}$, $\mathcal{PB}$, $\mathcal{STB}$ and $\mathcal{BCPB}$ for calculating CIs of the GPCI $\mathcal{C}_{py}$. Let $Y_1, Y_2, \ldots, Y_n$ be a random sample of size n drawn from exponential distribution with parameter ψ.

ALGORITHM:

- **Step 1:** From the given random sample of size n, I compute the MLE $\hat{\psi}$ of ψ. A bootstrap sample of size n is obtained from the original sample by putting $1/n$ as mass at each point, denoted by $Y_1^*, Y_2^*, \ldots, Y_n^*$.
- **Step 2:** We compute the MLE $\hat{\psi}^*$ of ψ as well as $\hat{\mathcal{C}}_{py}^*$ of $\mathcal{C}_{py}$. The m-th bootstrap estimator of $\mathcal{C}_{py}$ is computed as $\hat{\mathcal{C}}_{py}^{*(m)} = \hat{\mathcal{C}}_{py}$ $(Y_1^*, Y_2^*, \ldots, Y_n^*)$.

- **Step 3:** There are total number of n^n re-samples and I have calculate B values of $\hat{\mathcal{C}}^*_{py}$ from these re-samples. Each of these $\hat{\mathcal{C}}^*_{py}$ would be estimator of $\hat{\mathcal{C}}_{py}$. The arrangement of the entire collection in assending would constitute an empirical bootstrap distribution $\{\hat{\mathcal{C}}^{*(j)}_{py};\ j = 1, 2, \ldots, B\}$, will be denoted as $\hat{\mathcal{C}}^{*(1)}_{py} \leq \hat{\mathcal{C}}^{*(2)}_{py} \leq \cdots \leq \hat{\mathcal{C}}^{*(B)}_{py}$.

Here, in this study we considered $B = 1000$ bootstrap samples.

Standard Bootstrap ($\mathcal{SB}$) Confidence Interval

Let $\bar{\hat{\mathcal{C}}}^*_{py}$ and Se^* be the sample mean and sample standard deviation of $\{\hat{\mathcal{C}}^{*(j)}_{py};\ j = 1, 2, \ldots, B\}$, i.e.,

$$\bar{\hat{\mathcal{C}}}^*_{py} = \frac{1}{B}\sum_{j=1}^{B}\hat{\mathcal{C}}^{*(j)}_{py}$$

and

$$Se^* = \sqrt{\frac{1}{(B-1)}\sum_{j=1}^{B}\left(\hat{\mathcal{C}}^{*(j)}_{py} - \bar{\hat{\mathcal{C}}}^*_{py}\right)^2},$$

respectively. A $100(1-\alpha)\%$ $\mathcal{SB}$ CI of the index $\mathcal{C}_{py}$ is given by

$$\left\{\hat{\mathcal{C}}^*_{py} - z_{(\alpha/2)}.Se^*,\ \hat{\mathcal{C}}^*_{py} + z_{(\alpha/2)}.Se^*\right\}.$$

Percentile Bootstrap ($\mathcal{PB}$) Confidence Interval

Let $\hat{\mathcal{C}}^{*(\tau)}_{py}$ be the τ percentile of $\{\hat{\mathcal{C}}^{*(j)}_{py};\ j = 1, 2, \ldots, B\}$, i.e., $\hat{\mathcal{C}}^{*(\tau)}_{py}$ is such that

$$\frac{1}{B}\sum_{j=1}^{B} I\left(\hat{\mathcal{C}}^{*(j)}_{py} \leq \hat{\mathcal{C}}^{*(\tau)}_{py}\right) = \tau; \quad 0 < \tau < 1,$$

where, $I(\cdot)$ is the indicator function. A $100(1-\alpha)\%$ $\mathcal{PB}$ CI of the index $\mathcal{C}_{py}$ is given by

$$\left\{\hat{\mathcal{C}}^{*(B.(\alpha/2))}_{py},\ \hat{\mathcal{C}}^{*(B.(1-\alpha/2))}_{py}\right\},$$

where, $\hat{\mathcal{C}}^{*(r)}_{py}$ is the r-th ordered value on the list of the B bootstrap estimators of $\mathcal{C}_{py}$.

Student's t Bootstrap ($\mathcal{STB}$) Confidence Interval

Let S^* be the sample standard deviation of $\{\hat{\mathcal{C}}_{py}^{*(j)};\ j = 1, 2, \ldots, B\}$, i.e.,

$$S^* = \sqrt{\frac{1}{B}\sum_{j=1}^{B}\left(\hat{\mathcal{C}}_{py}^{*(j)} - \bar{\hat{\mathcal{C}}}_{py}^{*}\right)^2},$$

where,

$$\bar{\hat{\mathcal{C}}}_{py}^{*} = \frac{1}{B}\sum_{j=1}^{B}\hat{\mathcal{C}}_{py}^{*(j)}.$$

Also, let $\hat{t}^{*(\tau)}$ be the τ percentile of $\{\frac{\hat{\mathcal{C}}_{py}^{*(j)} - \hat{\mathcal{C}}_{py}}{S^*}\}; j = 1, 2, \ldots, B$, i.e., $\hat{t}^{*(\tau)}$ is such that

$$\frac{1}{B}\sum_{j=1}^{B} I\left(\frac{\hat{\mathcal{C}}_{py}^{*(j)} - \hat{\mathcal{C}}_{py}}{S^*} \leq \hat{t}^{*(\tau)}\right) = \tau; \quad 0 < \tau < 1,$$

where, $I(\cdot)$ is the indicator function. A $100(1-\alpha)\%$ $\mathcal{STB}$ CI of the index $\mathcal{C}_{py}$ is given by

$$\left\{\bar{\hat{\mathcal{C}}}_{py}^{*} - \hat{t}^{*(\alpha/2)}.S^*,\ \bar{\hat{\mathcal{C}}}_{py}^{*} + \hat{t}^{*(\alpha/2)}.S^*\right\}.$$

Bias-corrected Percentile Bootstrap ($\mathcal{BCPB}$) Confidence Interval

This approach has been introduced to correct for the potential bias. The first step is to locate the observed $\hat{\mathcal{C}}_{py}$ in the bootstrap order statistics $\hat{\mathcal{C}}_{py}^{*(1)} \leq \hat{\mathcal{C}}_{py}^{*(2)} \leq \cdots \leq \hat{\mathcal{C}}_{py}^{*(B)}$. Firstly, using the ordered distributions of $\{\hat{\mathcal{C}}_{py}^{*(j)};\ j = 1, 2, \ldots, B\}$, compute the probability

$$P_0 = \frac{1}{B}\sum_{j=1}^{B} I\left(\hat{\mathcal{C}}_{py}^{*(j)} \leq \hat{\mathcal{C}}_{py}\right),$$

where $I(\cdot)$ is the indicator function. Then, I have calculate $z_0 = \Phi^{-1}(P_0)$, where, $\Phi(\cdot)$ is the standard normal CDF and this value is used to calculate the probabilities P_l and P_u, defined as

$$P_l = \Phi\left(2z_0 - z_{(\alpha/2)}\right) \text{ and } P_u = \Phi\left(2z_0 + z_{(\alpha/2)}\right).$$

A $100(1-\alpha)\%$ $\mathcal{BCPB}$ CI of δ is given by

$$\left(\hat{\mathcal{C}}_{py}^{*(B.P_l)},\ \hat{\mathcal{C}}_{py}^{*(B.P_u)}\right).$$

where $\hat{\mathcal{C}}_{py}^{*(r)}$ is the r-th ordered value on the list of the B bootstrap estimators of $\mathcal{C}_{py}$.

4 Bayesian Estimation

The Bayesian estimation of the index $\mathcal{C}_{py}$ is presented in this section. Bayesian analysis is a logical technique to mix observed and prior data. Prior distributions are crucial in the development of the Bayes estimator(s). There is no simple approach for selecting priors for a specific situation. More information can be found in Arnold and Press (1983). We analyse Bayesian estimation on the assumption that the random variables have independent gamma priors in the premise of the foregoing arguments.Let $\psi \sim$ Gamma(a, b). Because the Gamma distribution is versatile, it can take on a variety of shapes depending on parameter values, making it a good candidate for model parameter priors. More information can be found in Kundu and Pradhan (2009). Thus, the prior distribution of ψ is

$$\pi(\psi) \;=\; \frac{b^a}{\Gamma(a)}\psi^{a-1}e^{-b\psi}; \quad \psi > 0, \tag{31}$$

where a, and b are the hyper-parameters and are assumed to be known. The posterior distribution of ψ under LnD, XgD, and AkD are given in Equs. (32), (33), and (34) respectively.

$$\begin{aligned} P_1(\psi \mid y) \;&=\; K_1^{-1}\left(\frac{\psi^2}{1+\psi}\right)^n \psi^{a-1} e^{-\psi(b+\sum\limits_{i=1}^{n} y_i)} \prod_{i=1}^{n}(1+y_i) \\ &=\; K_1^{-1}\psi^{2n+a-1} e^{-\psi(b+\sum\limits_{i=1}^{n} y_i)} (1+\psi)^{-n} \prod_{i=1}^{n}(1+y_i) \end{aligned} \tag{32}$$

where

$$K_1^{-1} = \int_0^\infty \left(\frac{\psi^2}{1+\psi}\right)^n \psi^{a-1} e^{-\psi(b+\sum\limits_{i=1}^{n} y_i)} \prod_{i=1}^{n}(1+y_i)\,d\psi$$

is the normalizing constant for LnD.

$$\begin{aligned} P_2(\psi \mid y) &= K_2^{-1}\left(\frac{\psi^2}{1+\psi}\right)^n \psi^{a-1} e^{-\psi(b+\sum\limits_{i=1}^{n} y_i)} \prod_{i=1}^{n}\left(1+\frac{\psi y_i^2}{2}\right) \\ &= K_2^{-1}\psi^{2n+a-1} e^{-\psi(b+\sum\limits_{i=1}^{n} y_i)} (1+\psi)^{-n} \prod_{i=1}^{n}\left(1+\frac{\psi y_i^2}{2}\right) \end{aligned} \tag{33}$$

where

$$K_2^{-1} = \int_0^\infty \left(\frac{\psi^2}{1+\psi}\right)^n \psi^{a-1} e^{-\psi(b+\sum\limits_{i=1}^{n} y_i)} \prod_{i=1}^{n}\left(1+\frac{\psi y_i^2}{2}\right) d\psi$$

is the normalizing constant for XgD.

$$\begin{aligned} P_3(\psi \mid y) &= K_3^{-1}\left(\frac{\psi^3}{2+\psi^2}\right)^n \psi^{a-1} e^{-\psi(b+\sum\limits_{i=1}^{n} y_i)} \prod_{i=1}^{n}(1+y_i^2) \\ &= K_3^{-1}\psi^{3n+a-1} e^{-\psi(b+\sum\limits_{i=1}^{n} y_i)} (2+\psi^2)^{-n} \prod_{i=1}^{n}(1+y_i^2) \end{aligned} \tag{34}$$

where

$$K_3^{-1} = \int_0^\infty \left(\frac{\psi^3}{2+\psi^2}\right)^n \psi^{a-1} e^{-\psi(b+\sum\limits_{i=1}^{n} y_i)} \prod_{i=1}^{n}(1+y_i^2) d\psi$$

is the normalizing constant for AkD. We use the SELF to obtain the Bayes estimates of $\mathcal{C}_{py}$. The expression of the loss functions, the corresponding Bayes estimator and posterior risk are provided in Table 1. Where d is the estimate of parameter ψ.

Notice that if we can obtain the posterior distribution of $\mathcal{C}_{py}$, then the Bayes estimate of $\mathcal{C}_{py}$ can be easily obtained, but the evaluation of the posterior distribution of $\mathcal{C}_{py}$ is quite tedious. Therefore, the Bayes estimate

Table 1 Bayes estimate under SELF and corresponding posterior risk

Loss function	Bayes estimator	Posterior risk
$L = \text{SELF} = (\psi - d)^2$	$E(\psi \mid \boldsymbol{y})$	$\text{Var}(\psi \mid \boldsymbol{y})$

under SELF of $\mathcal{C}_{py}$ for known U and L concerning LnD, XgD, and AkD, can be obtained by the Equations (35), (36) and (37), respectively.

$$\begin{aligned}
\hat{\mathcal{C}}_{py}^{LnD} &= E(\mathcal{C}_{py}|y) = \int_0^\infty \int_0^\infty \mathcal{C}_{py}\; P_1(\psi \mid y)\; d\psi \\
&= K^{-1} \int_0^\infty \psi^{2n+a-1} e^{-\psi(b+\sum_{i=1}^{n} y_i)} (1+\psi)^{-n} \prod_{i=1}^{n} (1+y_i) \\
&\quad \times \frac{1}{p_0} \left[\frac{1+\psi+\psi L}{1+\psi} e^{-\psi L} - \frac{1+\psi+\psi U}{1+\psi} e^{-\psi U} \right] d\psi \qquad (35)
\end{aligned}$$

$$\begin{aligned}
\hat{\mathcal{C}}_{py}^{XgD} &= E(\mathcal{C}_{py}|y) = \int_0^\infty \int_0^\infty \mathcal{C}_{py}\; P_2(\psi \mid y)\; d\psi \\
&= K^{-1} \int_0^\infty \psi^{2n+a-1} e^{-\psi(b+\sum_{i=1}^{n} y_i)} (1+\psi)^{-n} \prod_{i=1}^{n} \left(1+\frac{\psi y_i^2}{2}\right) \\
&\quad \times \frac{1}{p_0} \left[\frac{1+\psi+\psi L+\frac{\psi^2 L^2}{2}}{1+\psi} e^{-\psi L} \right. \\
&\quad \left. - \frac{1+\psi+\psi U+\frac{\psi^2 U^2}{2}}{1+\psi} e^{-\psi U} \right] d\psi \qquad (36)
\end{aligned}$$

$$\begin{aligned}
\hat{\mathcal{C}}_{py}^{AkD} &= E(\mathcal{C}_{py}|y) = \int_0^\infty \int_0^\infty \mathcal{C}_{py}\; P_3(\psi \mid y)\; d\psi \\
&= K^{-1} \int_0^\infty \psi^{3n+a-1} e^{-\psi(b+\sum_{i=1}^{n} y_i)} (2+\psi^2)^{-n} \prod_{i=1}^{n} (1+y_i^2) \\
&\quad \times \frac{1}{p_0} \left[\left(1+\frac{2\psi L+\psi^2 L^2}{1+\psi}\right) e^{-\psi L} \right. \\
&\quad \left. \times - \left(1+\frac{2\psi U+\psi^2 U^2}{2+\psi^2}\right) e^{-\psi U} \right] d\psi \qquad (37)
\end{aligned}$$

Equations (35), (36) and (37) do not yield any standard form due to the involvement of two integrals in the denominator as well as in the numerator. Hence, the analytical solution of the same is not possible. Therefore, one may use any Bayes computation technique to obtain the solutions. Here, we use one Bayes computation technique namely, the M-H algorithm, which is more frequently used to approximate the posterior expectations. The detailed description of this approximation is given below:

Metropolis-Hastings Algorithm

Here, we consider an algorithm suggested by Metropolis and Hastings to compute the Bayes estimate as well as the credible interval of the index based on generated posterior samples. In this algorithm, samples are generated from the fully conditional posterior densities using an appropriate proposal distribution. The generated samples from the full conditional distribution are collected using the acceptance-rejection principle. For more details about this algorithm, the reader may refer to the articles by Metropolis et al. (1953), Smith and Robert (1993), and many others. To implement the M-H algorithm, the full conditional density of ψ under LnD can be written as;

$$P_1(\psi \mid y) \propto \psi^{2n+a-1} e^{-\psi(b+\sum_{i=1}^{n} y_i)} (1+\psi)^{-n} \prod_{i=1}^{n} (1+y_i) \tag{38}$$

The following algorithm may be used to extract the samples from $P_1(\psi \mid y)$.

1. Set the initial guess value $\{\psi^{(0)}\}$.
2. Begin with $r = 1$.
3. Generate a new sample for ψ from the respective conditional posterior densities by choosing any arbitrary proposal distribution as follows: $\psi^{(r)} \sim P_1(y \mid \psi^{(r-1)})$
4. Repeat step 2-3 for all $r = 1, 2, 3, \ldots, K(= 10000)$ times and obtain posterior samples of size K for parameters ψ.
5. Using the above sequences obtained in step 4, we can obtain the sequence $\mathcal{C}_{py}^{r}$.

   After obtaining the posterior samples, the Bayes estimate of $\mathcal{C}_{py}$ under SELF is obtained as

$$\hat{\mathcal{C}}_{py}^{LnD} = E(\mathcal{C}_{py} \mid y) \approx \frac{1}{K-K_0} \sum_{r=K_0+1}^{K} \mathcal{C}_{py}^{r} \tag{39}$$

Similarly we can get the Bayes estimate of $\mathcal{C}_{py}$ under SELF for XgD and AkD respectively, as follows

$$\hat{\mathcal{C}}_{py}^{XgD} = E(\mathcal{C}_{py} \mid y) \approx \frac{1}{K-K_0} \sum_{r=K_0+1}^{K} \mathcal{C}_{py}^{r} \tag{40}$$

$$\hat{\mathcal{C}}_{py}^{AkD} = E(\mathcal{C}_{py} \mid y) \approx \frac{1}{K-K_0} \sum_{r=K_0+1}^{K} \mathcal{C}_{py}^{r} \tag{41}$$

where $K_0 = 500$ is the burn-in-period of Markov Chain.

6. Chen and Shao (1999) suggested the algorithm by which we can get the $100(1-\alpha)\%$ HPD credible interval for the index $\mathcal{C}_{py}$ under considered models.

5 Simulation and Discussions

Here, we have carried out a Monte Carlo simulation study to assess the performances of the GPCIs $\mathcal{C}_{py}$ under-considered models (LnD, XgD, AkD) using classical methods (MLE, LSE, WLSE, MPSE) and the Bayesian method of estimation. The classical estimators' performances are evaluated in terms of MSEs, whereas the Bayes estimators are evaluated in terms of simulated risk. Besides, we have constructed BCIs ($\mathcal{SB}$, $\mathcal{PB}$, $\mathcal{STB}$, $\mathcal{BCPB}$) for classical methods of estimation and HPD credible intervals for the Bayesian method. The performances of different CIs (BCIs and HPD) are assessed based on their estimated $\mathcal{AW}$s. "$\mathcal{AW}$" is the ratio of the sum of the differences between the upper and lower specification limits to the number of trials K and a lower $\mathcal{AW}$ indicates better performance. we consider the sample sizes n = 10, 20, 30, 50 and 100, for parameter (ψ) = 0.25, 0.75, 1.0, 1.25 with (L, U) = (0.1, 6) and $p_0 = 0.95$, respectively. For each design, samples of each size n are drawn from the original sample and replicated 3,000 times. For Bayesian computation, we have considered the hyper-parameter values of the informative prior for comparing the Bayes estimates under the considered models. We have chosen the hyper-parameter values arbitrarily as $(a, b) = (0.06, 0.25), (0.56, 0.75), (1, 1), (1.56, 1.25)$ for different sets of parameter values.

The estimate and corresponding MSEs of GPCI $\mathcal{C}_{py}$ for LnD, XgD and AkD are obtained through classical methods of estimation and reported in Tables 2, 3, and 4, respectively. BCIs of GPCI $\mathcal{C}_{py}$ for considered classical

Table 2 True values and estimated values of $\mathcal{C}_{py}$ by different methods of estimation along with their MSEs for LnD

n		$\mathcal{C}_{py}$=0.8774483,		ψ=0.5		$\mathcal{C}_{py}$=0.976662,		ψ=0.75	
		MLE	LSE	WLSE	MPSE	MLE	LSE	WLSE	MPSE
10	Est.	0.879789	0.865363	0.868285	0.855203	0.964209	0.983523	0.958614	0.958619
	MSE	0.005885	0.008642	0.007083	0.005586	0.002215	0.089940	0.001800	0.001748
20	Est.	0.858109	0.851396	0.853165	0.840554	0.970680	0.967701	0.967992	0.964575
	MSE	0.004977	0.006025	0.005685	0.004565	0.000495	0.000693	0.000677	0.000396
30	Est.	0.878808	0.876160	0.876407	0.866174	0.971949	0.969396	0.969704	0.967377
	MSE	0.002393	0.002636	0.002467	0.002165	0.000409	0.000454	0.000426	0.000365
50	Est.	0.876891	0.874232	0.874623	0.868169	0.972035	0.972308	0.971977	0.968636
	MSE	0.001509	0.001750	0.001623	0.001491	0.000137	0.000216	0.000201	0.000102
100	Est.	0.877121	0.874574	0.875038	0.872000	0.975294	0.974632	0.974821	0.973517
	MSE	0.000697	0.000837	0.000777	0.000671	0.000580	0.000103	0.000092	0.000018
n		$\mathcal{C}_{py}$=0.9896466,		ψ=1		$\mathcal{C}_{py}$=0.9780293,		ψ=1.25	
		MLE	LSE	WLSE	MPSE	MLE	LSE	WLSE	MPSE
10	Est.	0.978130	0.978673	0.976522	0.978587	0.968040	0.968913	0.968853	0.973779
	MSE	0.000493	0.005604	0.000511	0.000672	0.000462	0.002232	0.000804	0.000428
20	Est.	0.984350	0.983335	0.983741	0.984690	0.973546	0.973813	0.974149	0.977258
	MSE	0.000090	0.000131	0.000117	0.000081	0.000258	0.000293	0.000271	0.000182
30	Est.	0.985879	0.985132	0.985448	0.986250	0.974904	0.975145	0.975368	0.977817
	MSE	0.000046	0.000064	0.000056	0.000040	0.000154	0.000192	0.000176	0.000115
50	Est.	0.987334	0.986917	0.987122	0.987655	0.976161	0.976158	0.976299	0.978245
	MSE	0.000019	0.000025	0.000022	0.000015	0.000085	0.000106	0.000098	0.000070
100	Est.	0.988560	0.988400	0.988485	0.988769	0.977240	0.977284	0.977348	0.978470
	MSE	0.000005	0.000007	0.000006	0.000004	0.000040	0.000052	0.000047	0.000036

methods are provided in Tables 5, 6, and 7 for LnD, XgD and AkD, respectively. For all the models, Bayes estimates with risk and HPD credible interval through M-H algorithm are given in Tables 8 and 9. From first three tables, we observed that LnD performs better than XgD and AkD in terms of MSEs under considered classical methods and for considered parameter setups except for $\psi = 1.25$. MPSE gives the smallest MSEs among all classical methods for almost all the considered setups and this trend is similar in all considered models. Analysis of Tables 5, 6, and 7 depicts that among all BCIs $\mathcal{STB}$ gives the least $\mathcal{AW}$ under all classical methods and for all models. Besides, MPSE performs better in calculating the $\mathcal{AW}$ of BCIs in all models. Among considered models LnD gives batter $\mathcal{AW}$ for all most all the considered parameter setups except for $\psi = 1.25$. In Bayesian estimation using the M-H algorithm, LnD performs better as compared to Xgd and AkD in terms of their smaller average risks, and the HPD credible interval is also small for LnD as compared to other models for all parameter setups. From

Table 3 True values and estimated values of $\mathcal{C}_{py}$ by different methods of estimation along with their MSEs for XgD

n		$\mathcal{C}_{py}$=0.7210604,		ψ=0.5		$\mathcal{C}_{py}$=0.9105752,		ψ=0.75	
		MLE	LSE	WLSE	MPSE	MLE	LSE	WLSE	MPSE
10	Est.	0.729966	0.714715	0.713854	0.693307	0.903001	0.890752	0.891013	0.880434
	MSE	0.014786	0.015694	0.015036	0.012326	0.004911	0.005714	0.005512	0.004085
20	Est.	0.728137	0.719640	0.719848	0.704688	0.905420	0.899643	0.899999	0.890818
	MSE	0.007981	0.008916	0.008364	0.007333	0.002625	0.002965	0.002800	0.002389
30	Est.	0.727594	0.725333	0.725641	0.709470	0.906755	0.907548	0.906732	0.895123
	MSE	0.004996	0.005767	0.005273	0.004665	0.000866	0.001503	0.000912	0.000864
50	Est.	0.722135	0.719436	0.719514	0.709635	0.909521	0.903733	0.905410	0.902261
	MSE	0.002918	0.003425	0.003162	0.002701	0.000635	0.000915	0.000794	0.000575
100	Est.	0.721463	0.720774	0.720243	0.714225	0.909737	0.908302	0.908615	0.905389
	MSE	0.001457	0.001679	0.001543	0.001319	0.000439	0.000591	0.000534	0.000419
n		$\mathcal{C}_{py}$=0.9685448,		ψ=1		$\mathcal{C}_{py}$=0.9739773,		ψ=1.25	
		MLE	LSE	WLSE	MPSE	MLE	LSE	WLSE	MPSE
10	Est.	0.956544	0.949060	0.949947	0.947833	0.962388	0.960015	0.960755	0.963296
	MSE	0.000768	0.001448	0.001333	0.000626	0.000425	0.000657	0.000601	0.000389
20	Est.	0.962373	0.958862	0.959469	0.957306	0.968104	0.966722	0.967289	0.969029
	MSE	0.000409	0.000517	0.000470	0.000349	0.000123	0.000192	0.000167	0.000096
30	Est.	0.964096	0.961881	0.962379	0.960412	0.970038	0.969356	0.969712	0.970876
	MSE	0.000193	0.000293	0.000263	0.000132	0.000060	0.000082	0.000072	0.000043
50	Est.	0.965869	0.964440	0.964783	0.963392	0.971533	0.970948	0.971211	0.972213
	MSE	0.000102	0.000140	0.000129	0.000083	0.000028	0.000041	0.000036	0.000019
100	Est.	0.967249	0.966848	0.966984	0.965821	0.972791	0.972544	0.972685	0.973224
	MSE	0.000043	0.000056	0.000051	0.000028	0.000010	0.000014	0.000012	0.000007

Tables 2 to 9, it has been observed that as the sample sizes increase, the MSEs, and risks of all the estimators are decrease, which verifies the consistency of the estimators that we have considered. Besides, the $\mathcal{AW}$s of BCIs and HPD credible intervals also decreased as we increased the sample size.

6 Data Analysis

In this section, we consider two real data sets and analyzed for illustrative purposes. Descriptive statistics of the considered data sets are displayed in Table 10. First, using the goodness of fit test, we verify whether the given data sets confirm that they belong to the LnD, XgD, and AkD. Results of the goodness of fit test are reported in Table 11. From Table 11, it is observed that the p-values for both the data sets are much higher than the level of significance (0.05), which indicates that the considered data sets are suitable for the considered model.

Table 4 True values and estimated values of $\mathcal{C}_{py}$ by different methods of estimation along with their MSEs for AkD

n		$\mathcal{C}_{py}$=0.6451183,		ψ=0.5		$\mathcal{C}_{py}$=0.8907082,		ψ=0.75	
		MLE	LSE	WLSE	MPSE	MLE	LSE	WLSE	MPSE
10	Est.	0.665227	0.652336	0.652887	0.594764	0.881376	0.872612	0.872651	0.833047
	MSE	0.015521	0.016210	0.015715	0.014408	0.006537	0.006923	0.006747	0.005484
20	Est.	0.627245	0.617239	0.618919	0.582421	0.889076	0.884923	0.885180	0.861044
	MSE	0.008680	0.009645	0.009381	0.008176	0.002748	0.003307	0.003152	0.002358
30	Est.	0.649798	0.647868	0.647036	0.617526	0.887893	0.885007	0.885383	0.867529
	MSE	0.004458	0.005178	0.004889	0.004307	0.001974	0.002366	0.002235	0.001892
50	Est.	0.646021	0.644475	0.644294	0.624784	0.889751	0.887847	0.888168	0.876628
	MSE	0.002906	0.003442	0.003202	0.002371	0.001227	0.001432	0.001347	0.001199
100	Est.	0.648702	0.650901	0.650566	0.636540	0.890053	0.889191	0.889390	0.882843
	MSE	0.001590	0.001980	0.001787	0.001573	0.000582	0.000682	0.000634	0.000488
n		$\mathcal{C}_{py}$=0.9747761,		ψ=1		$\mathcal{C}_{py}$=0.9859814,		ψ=1.25	
		MLE	LSE	WLSE	MPSE	MLE	LSE	WLSE	MPSE
10	Est.	0.966257	0.962349	0.962575	0.948346	0.975280	0.974110	0.972375	0.973278
	MSE	0.000931	0.001073	0.001035	0.000924	0.000590	0.001503	0.002012	0.000585
20	Est.	0.969145	0.966658	0.967025	0.958633	0.980810	0.980119	0.979125	0.981273
	MSE	0.000408	0.000574	0.000538	0.000338	0.000090	0.000117	0.001085	0.000084
30	Est.	0.970646	0.968992	0.969319	0.963343	0.982414	0.982034	0.981788	0.983060
	MSE	0.000273	0.000358	0.000334	0.000234	0.000048	0.000062	0.000457	0.000036
50	Est.	0.972129	0.971183	0.971401	0.967524	0.983865	0.983576	0.983716	0.984473
	MSE	0.000151	0.000186	0.000173	0.000122	0.000019	0.000024	0.000022	0.000012
100	Est.	0.973401	0.972908	0.973051	0.970956	0.984961	0.984843	0.984905	0.985380
	MSE	0.000067	0.000080	0.000074	0.000044	0.000006	0.000007	0.000006	0.000003

- **Data set I:** The data set represents the waiting time (in minutes) before customer service in a bank the detailed description of the data set is mentioned in Ghitany et al. (2008). Here, we assume that the upper and lower specification limits L = 1 and U = 35.1 (each measurement in minutes), respectively.
- **Data set II:** The second data set is regarding the first failure time (time in months) of 20 electric carts used for internal transformation and delivery in a large manufacturing facility. This data set discussed by Zimmer et al. (1998) for the Burr XII reliability analysis. Here, we assume that the upper and lower specification limits L = 0.95 and U = 52.1 (each measurement in minutes), respectively.

For the considered data sets, we have calculated the point estimates of GPCI $\mathcal{C}_{py}$ using different classical estimation methods and the Bayesian estimation method. The classical estimates of the considered index are reported in Table 12 and the Bayes estimates (point and interval) of GPCI $\mathcal{C}_{py}$ under

Table 5 True values and $\mathcal{AW}s$ of $\mathcal{C}_{py}$ of BCIs for LnD

$\mathcal{C}_{py}$ ψ	n	MLE				LSE			
		$\mathcal{SB}$	$\mathcal{PB}$	$\mathcal{STB}$	$\mathcal{BCPB}$	$\mathcal{SB}$	$\mathcal{PB}$	$\mathcal{STB}$	$\mathcal{BCPB}$
	10	0.294136	0.280690	0.209344	0.291864	0.656123	0.321227	0.207201	0.414685
0.877448	20	0.210582	0.205766	0.166374	0.211112	0.243901	0.236562	0.182931	0.233462
0.5	30	0.179497	0.177130	0.150699	0.180705	0.200585	0.194900	0.157803	0.193299
	50	0.145064	0.143915	0.128103	0.145269	0.154407	0.152830	0.129211	0.150823
	100	0.105432	0.104884	0.097071	0.105360	0.115049	0.114632	0.102558	0.114347
	10	0.135649	0.126159	0.047982	0.115180	0.186165	0.172515	0.069833	0.142938
0.976662	20	0.089950	0.084236	0.040070	0.082346	0.138575	0.102258	0.047206	0.090716
0.75	30	0.073276	0.069389	0.037925	0.069920	0.077388	0.072993	0.033078	0.066440
	50	0.051911	0.049812	0.029438	0.050939	0.059311	0.056615	0.030896	0.054296
	100	0.035199	0.034268	0.023810	0.035099	0.036388	0.035104	0.021778	0.034802
	10	0.093305	0.086358	0.022290	0.057988	0.115793	0.105702	0.022045	0.074261
0.989647	20	0.047465	0.043928	0.010436	0.031123	0.088019	0.056408	0.014554	0.040293
1	30	0.032484	0.030079	0.008168	0.021752	0.040860	0.037452	0.010238	0.029139
	50	0.021916	0.017863	0.004590	0.012408	0.022963	0.021204	0.005139	0.014993
	100	0.010215	0.009458	0.002727	0.006986	0.013411	0.012480	0.004046	0.010123
	10	0.102690	0.096411	0.033154	0.072400	0.123670	0.114753	0.037203	0.090518
0.978029	20	0.066462	0.063186	0.031317	0.057623	0.097819	0.065840	0.028744	0.062803
1.25	30	0.047723	0.045609	0.024369	0.042230	0.055094	0.052678	0.027689	0.051649
	50	0.035140	0.034105	0.020892	0.032773	0.037576	0.035928	0.020731	0.036242
	100	0.025194	0.024784	0.018293	0.024416	0.028522	0.028001	0.020407	0.028377
$\mathcal{C}_{py}$ ψ	n	WLSE				MPSE			
		$\mathcal{SB}$	$\mathcal{PB}$	$\mathcal{STB}$	$\mathcal{BCPB}$	$\mathcal{SB}$	$\mathcal{PB}$	$\mathcal{STB}$	$\mathcal{BCPB}$
	10	0.310544	0.295794	0.203689	0.282632	0.285017	0.273379	0.202178	0.293050
0.877448	20	0.229858	0.224253	0.171414	0.222131	0.208656	0.203425	0.160410	0.203876
0.5	30	0.201329	0.198203	0.166035	0.196747	0.175773	0.172703	0.144598	0.179581
	50	0.153786	0.151717	0.131460	0.151669	0.143901	0.142529	0.121321	0.145268
	100	0.111167	0.110506	0.099566	0.110349	0.100833	0.101413	0.088174	0.103117
	10	0.159903	0.148093	0.049612	0.116259	0.131197	0.118116	0.043433	0.114271
0.976662	20	0.101563	0.094851	0.039842	0.084764	0.081318	0.081223	0.039787	0.078596
0.75	30	0.076629	0.072511	0.034718	0.067302	0.068384	0.064629	0.034083	0.068826
	50	0.058623	0.056142	0.032215	0.054542	0.050628	0.048607	0.026266	0.049518
	100	0.036052	0.034921	0.022859	0.034456	0.033529	0.031570	0.023427	0.034690
	10	0.116290	0.106394	0.026012	0.079890	0.090587	0.078179	0.021037	0.056182
0.989647	20	0.057771	0.053797	0.014953	0.040517	0.044052	0.040166	0.010065	0.031055
1	30	0.037875	0.035132	0.008797	0.024993	0.030070	0.002905	0.006557	0.019737
	50	0.021367	0.019675	0.004449	0.013659	0.020338	0.016791	0.004503	0.011977
	100	0.011966	0.011194	0.003378	0.008930	0.009796	0.009039	0.002362	0.006800
	10	0.116114	0.106735	0.032383	0.087482	0.090236	0.083952	0.024682	0.070959
0.978029	20	0.069914	0.066120	0.030636	0.061823	0.055009	0.051701	0.026083	0.056654
1.25	30	0.056191	0.053826	0.030470	0.053858	0.039090	0.037047	0.021346	0.042029
	50	0.039595	0.038334	0.024173	0.038526	0.030270	0.029027	0.019838	0.030790
	100	0.026729	0.026320	0.019296	0.026591	0.023065	0.022606	0.017519	0.023049

Table 6 True values and $\mathcal{AW}s$ of $\mathcal{C}_{py}$ of BCIs for XgD

$\mathcal{C}_{py}$ ψ	n	MLE				LSE			
		$\mathcal{SB}$	$\mathcal{PB}$	$\mathcal{STB}$	$\mathcal{BCPB}$	$\mathcal{SB}$	$\mathcal{PB}$	$\mathcal{STB}$	$\mathcal{BCPB}$
	10	0.404213	0.396618	0.341336	0.372246	0.464587	0.448132	0.449843	0.432402
0.721060	20	0.306730	0.302914	0.298811	0.299697	0.348492	0.343875	0.352192	0.337741
0.5	30	0.256145	0.253767	0.263676	0.253334	0.289339	0.285637	0.304868	0.284427
	50	0.202482	0.201293	0.219669	0.204139	0.222909	0.221130	0.206987	0.218780
	100	0.143889	0.143036	0.139454	0.142729	0.159815	0.159838	0.155933	0.156733
	10	0.230394	0.214949	0.137809	0.244377	0.284017	0.261854	0.179937	0.270438
0.910575	20	0.166718	0.160138	0.114500	0.164097	0.191347	0.181943	0.139160	0.192199
0.75	30	0.151999	0.149267	0.112405	0.142428	0.152699	0.147826	0.114533	0.145574
	50	0.108212	0.106523	0.104902	0.123842	0.123989	0.122205	0.100959	0.110834
	100	0.078546	0.078197	0.071296	0.080435	0.095715	0.095071	0.074667	0.089110
	10	0.100545	0.091774	0.028866	0.075334	0.176356	0.162865	0.074564	0.140182
0.968545	20	0.072646	0.066961	0.023910	0.075030	0.107431	0.099626	0.034733	0.079362
1	30	0.054667	0.050676	0.021592	0.048380	0.074976	0.069483	0.026981	0.061665
	50	0.035131	0.033110	0.017676	0.035334	0.053742	0.050940	0.024086	0.049720
	100	0.027000	0.025704	0.015135	0.024920	0.025823	0.024285	0.013000	0.028615
	10	0.113921	0.104872	0.034864	0.058311	0.124001	0.111816	0.033211	0.127577
0.973977	20	0.055423	0.050783	0.018254	0.043441	0.058817	0.053841	0.013526	0.047762
1.25	30	0.031787	0.030295	0.009966	0.026078	0.046213	0.042261	0.013871	0.036187
	50	0.020585	0.019022	0.006952	0.019088	0.025613	0.023488	0.008151	0.017135
	100	0.011518	0.010725	0.004295	0.010787	0.013442	0.012402	0.004393	0.012015
$\mathcal{C}_{py}$ ψ	n	WLSE				MPSE			
		$\mathcal{SB}$	$\mathcal{PB}$	$\mathcal{STB}$	$\mathcal{BCPB}$	$\mathcal{SB}$	$\mathcal{PB}$	$\mathcal{STB}$	$\mathcal{BCPB}$
	10	0.460088	0.446721	0.434481	0.427817	0.400653	0.383063	0.337199	0.369410
0.721060	20	0.344096	0.339964	0.391265	0.330106	0.297349	0.293130	0.252886	0.296504
0.5	30	0.273240	0.271835	0.238852	0.260739	0.255607	0.253129	0.251097	0.252395
	50	0.217184	0.216057	0.222478	0.214956	0.194604	0.192556	0.194341	0.191107
	100	0.157843	0.157920	0.164907	0.157340	0.147023	0.140212	0.124748	0.141724
	10	0.302805	0.283702	0.197969	0.301453	0.219958	0.203534	0.123130	0.242078
0.910575	20	0.212119	0.204998	0.123383	0.164024	0.154204	0.158711	0.100125	0.151573
0.75	30	0.162883	0.158423	0.136942	0.170730	0.148111	0.143458	0.105396	0.140124
	50	0.125488	0.123054	0.110634	0.130880	0.091369	0.099841	0.085208	0.109574
	100	0.098558	0.098506	0.100252	0.092195	0.061364	0.068010	0.056557	0.067917
	10	0.173100	0.157255	0.052056	0.146356	0.097198	0.090542	0.028151	0.074352
0.968545	20	0.098569	0.091090	0.039812	0.078501	0.071171	0.065810	0.023385	0.071768
1	30	0.054262	0.049740	0.026511	0.049619	0.045541	0.040886	0.020895	0.045946
	50	0.052109	0.048848	0.024247	0.046091	0.034406	0.032117	0.013219	0.034688
	100	0.029045	0.027754	0.018093	0.029052	0.026965	0.024875	0.014228	0.023341
	10	0.115728	0.102848	0.033623	0.083063	0.111466	0.101528	0.028261	0.048911
0.973977	20	0.053489	0.049699	0.011128	0.034469	0.046053	0.045547	0.017531	0.035451
1.25	30	0.036628	0.034693	0.008564	0.030869	0.030033	0.029148	0.005871	0.025056
	50	0.020442	0.019054	0.005336	0.018296	0.019779	0.018395	0.006049	0.018692
	100	0.011054	0.009930	0.003267	0.012478	0.009584	0.008882	0.002277	0.004374

Table 7 True values and $\mathcal{AW}s$ of $\mathcal{C}_{py}$ of BCIs for AkD

$\mathcal{C}_{py}$ ψ	n	MLE				LSE			
		$\mathcal{SB}$	$\mathcal{PB}$	$\mathcal{STB}$	$\mathcal{BCPB}$	$\mathcal{SB}$	$\mathcal{PB}$	$\mathcal{STB}$	$\mathcal{BCPB}$
	10	0.437687	0.430605	0.411914	0.415991	0.461418	0.452105	0.402186	0.428910
0.645118	20	0.330898	0.327374	0.332497	0.324617	0.354265	0.349750	0.328684	0.345806
0.5	30	0.279391	0.278132	0.295479	0.276766	0.302593	0.300121	0.329704	0.298886
	50	0.215426	0.214608	0.217570	0.214397	0.233573	0.232874	0.236927	0.231654
	100	0.153630	0.153512	0.146728	0.152530	0.164612	0.163342	0.150733	0.162116
	10	0.266976	0.250841	0.204355	0.285924	0.323048	0.307740	0.193237	0.261823
0.890708	20	0.209480	0.205526	0.137988	0.180488	0.219456	0.212140	0.183338	0.238014
0.75	30	0.169684	0.167224	0.148357	0.175352	0.198359	0.196279	0.160253	0.193397
	50	0.134632	0.132693	0.112369	0.131351	0.145054	0.143826	0.112019	0.135153
	100	0.093588	0.093092	0.090389	0.098186	0.099968	0.100422	0.089721	0.091989
	10	0.115747	0.106237	0.045426	0.108971	0.153401	0.141536	0.054918	0.150901
0.974776	20	0.103980	0.098063	0.036447	0.067112	0.100915	0.094895	0.041952	0.085862
1	30	0.056523	0.052851	0.025361	0.057322	0.075928	0.071461	0.032226	0.062132
	50	0.049763	0.047807	0.022174	0.035941	0.053144	0.050744	0.027869	0.049205
	100	0.034374	0.033507	0.022178	0.031629	0.036033	0.035189	0.022769	0.032929
	10	0.081450	0.073808	0.018443	0.051332	0.112037	0.100418	0.036216	0.104392
0.985981	20	0.047723	0.044160	0.013888	0.035435	0.052700	0.048457	0.018813	0.044577
1.25	30	0.024061	0.021944	0.004342	0.013169	0.041736	0.038384	0.015630	0.028089
	50	0.016988	0.015716	0.003935	0.012054	0.017449	0.015978	0.003560	0.010694
	100	0.009086	0.008278	0.002493	0.006470	0.011093	0.010151	0.002729	0.007426
$\mathcal{C}_{py}$ ψ	n	WLSE				MPSE			
		$\mathcal{SB}$	$\mathcal{PB}$	$\mathcal{STB}$	$\mathcal{BCPB}$	$\mathcal{SB}$	$\mathcal{PB}$	$\mathcal{STB}$	$\mathcal{BCPB}$
	10	0.476791	0.467952	0.468416	0.450657	0.427025	0.425572	0.403884	0.415798
0.645118	20	0.352671	0.348144	0.362391	0.344774	0.327929	0.322354	0.300892	0.327757
0.5	30	0.288292	0.285888	0.276887	0.282797	0.278749	0.274946	0.205127	0.266286
	50	0.223318	0.222663	0.212974	0.219545	0.213691	0.212494	0.165830	0.209318
	100	0.161620	0.161200	0.163585	0.160669	0.150806	0.150429	0.134732	0.146557
	10	0.316434	0.301652	0.213860	0.297511	0.239781	0.238908	0.194207	0.264864
0.890708	20	0.220467	0.214655	0.174164	0.218879	0.200038	0.200559	0.129140	0.171952
0.75	30	0.171470	0.169755	0.146188	0.143276	0.160071	0.158180	0.120468	0.163569
	50	0.136668	0.135182	0.117604	0.136248	0.122186	0.121402	0.107219	0.124179
	100	0.103155	0.102816	0.064773	0.085772	0.083885	0.083734	0.069758	0.090797
	10	0.075981	0.069321	0.010496	0.013689	0.109536	0.100336	0.039607	0.094399
0.974776	20	0.105981	0.096204	0.149310	0.097202	0.096568	0.090410	0.034790	0.058332
1	30	0.096203	0.093554	0.114821	0.054171	0.048971	0.048598	0.024529	0.047855
	50	0.076862	0.075845	0.027831	0.037932	0.044006	0.042025	0.021365	0.034068
	100	0.034664	0.033649	0.024008	0.034496	0.033836	0.033161	0.022179	0.031264
	10	0.080546	0.073575	0.020598	0.013777	0.079109	0.070205	0.009976	0.032750
0.985981	20	0.069279	0.065396	0.123183	0.030402	0.045865	0.040532	0.010402	0.023541
1.25	30	0.046304	0.042530	0.026635	0.045790	0.020221	0.018201	0.004187	0.012621
	50	0.044698	0.044740	0.089913	0.012600	0.016837	0.015361	0.002237	0.008470
	100	0.021210	0.020895	0.039214	0.013721	0.008830	0.008271	0.002185	0.005409

Table 8 True and Bayes estimate of $\mathcal{C}_{py}$ along with the Risk under SELF through M-H algorithm for LnD, XgD, and AkD

Model		Estimate (Est.) and Risk of $\mathcal{C}_{py}$ through M-H algorithm							
	$\mathcal{C}_{py}$	0.8774483		0.976662		0.989647		0.978029	
	ψ	0.5		0.75		1		1.25	
	n	Est.	Risk	Est.	Risk	Est.	Risk	Est.	Risk
LnD	10	0.853024	0.004214	0.948373	0.004177	0.969128	0.000240	0.961106	0.000452
	20	0.864481	0.003102	0.960922	0.001487	0.979044	0.000110	0.971526	0.000264
	30	0.874767	0.001541	0.962843	0.001110	0.982419	0.000076	0.972255	0.000065
	50	0.869895	0.001423	0.971615	0.000170	0.985448	0.000009	0.974809	0.000129
	100	0.873286	0.000923	0.973450	0.000005	0.987620	0.000006	0.976895	0.000038
	$\mathcal{C}_{py}$	0.7210604		0.9105752		0.968545		0.973977	
	ψ	0.5		0.75		1		1.25	
	n	Est.	Risk	Est.	Risk	Est.	Risk	Est.	Risk
XgD	10	0.756474	0.009039	0.926176	0.003752	0.953478	0.000297	0.942043	0.000613
	20	0.767116	0.005587	0.930856	0.001934	0.963577	0.000290	0.952245	0.000238
	30	0.790385	0.003173	0.937375	0.001010	0.967313	0.000066	0.950271	0.000069
	50	0.785451	0.002058	0.945529	0.000096	0.969858	0.000010	0.954306	0.000046
	100	0.789346	0.000894	0.948281	0.000199	0.972459	0.000004	0.955171	0.000043
	$\mathcal{C}_{py}$	0.6451183		0.8907082		0.974776		0.985981	
	ψ	0.5		0.75		1		1.25	
	n	Est.	Risk	Est.	Risk	Est.	Risk	Est.	Risk
AkD	10	0.565347	0.017604	0.796856	0.011609	0.920499	0.004731	0.956472	0.000569
	20	0.559024	0.008279	0.819741	0.007189	0.938336	0.001272	0.971138	0.000175
	30	0.540392	0.006021	0.816193	0.004546	0.941741	0.000420	0.976528	0.000077
	50	0.553668	0.003820	0.816866	0.002636	0.941253	0.001816	0.979522	0.000021
	100	0.559860	0.001870	0.824001	0.001318	0.948663	0.000237	0.983446	0.000006

Table 9 True value of $\mathcal{C}_{py}$ along with HPD Interval in terms of $\mathcal{AW}s$ for LnD, XgD and AkD

HPD interval of $\mathcal{C}_{py}$ through M-H algorithm						
Model	n	$\mathcal{C}_{py}$	0.877448	0.976662	0.989647	0.978029
		ψ	0.5	0.75	1	1.25
LnD	10	HPD ($\mathcal{AW}s$)	0.286503	0.128196	0.069984	0.071591
	20		0.216304	0.079287	0.034720	0.044250
	30		0.177423	0.068828	0.023888	0.038950
	50		0.145669	0.044250	0.014169	0.030460
	100		0.103922	0.032680	0.007353	0.022453
	n	$\mathcal{C}_{py}$	0.721060	0.910575	0.968545	0.973977
		ψ	0.5	0.75	1	1.25
XgD	10	HPD ($\mathcal{AW}s$)	0.331779	0.131199	0.061547	0.072174
	20		0.260186	0.101713	0.034209	0.046301
	30		0.205779	0.081814	0.023311	0.042738
	50		0.164615	0.058323	0.015419	0.033091
	100		0.118632	0.041921	0.007601	0.025168
	n	$\mathcal{C}_{py}$	0.645118	0.890708	0.974776	0.985981
		ψ	0.5	0.75	1	1.25
AkD	10	HPD ($\mathcal{AW}s$)	0.491227	0.380292	0.195269	0.106567
	20		0.366641	0.287847	0.127084	0.054387
	30		0.302958	0.246231	0.109191	0.036017
	50		0.239482	0.197394	0.094328	0.023970
	100		0.172408	0.140885	0.065415	0.011301

Table 10 Descriptive Statistics for the considered data sets

Data Sets	Minimum	$Q1$	median	mean	$Q3$	Maximum	Sd	CS	CK
I	0.8	4.675	8.1	9.877	13.02	38.5	7.236	1.472	5.54
II	0.9	4.725	10.75	14.68	20.12	53	13.663	1.348	4.279

Table 11 Goodness of fit summary for considered data set

Data Sets	Model	-Log Likelihood	AIC	BIC	K.S Statistics	K.S (p-value)
I	LnD	319.0374	640.0748	642.6800	0.0677	0.7495
	XgD	132.7684	267.5367	270.1419	0.0625	0.8297
	AkD	320.9646	643.9292	646.5344	0.1003	0.2672
II	LnD	74.5745	151.1490	152.1448	0.1254	0.8736
	XgD	75.9128	153.8256	154.8214	0.1753	0.5146
	AkD	79.1776	160.3552	161.3510	0.2071	0.3130

Table 12 Estimates of GPCIs $\mathcal{C}_{py}$ using different methods of estimation

Data Sets	Model	$\hat{\psi}$	$\hat{C}_{py}$			
			MLE	LSE	WLSE	MPSE
I	LnD	0.186571	1.000987	1.001030	1.001154	0.015165
	XgD	0.263407	0.995442	0.993535	0.994805	0.001645
	AkD	0.295277	1.035844	1.033791	1.034129	0.000834
II	LnD	0.128526	1.023422	1.023643	1.023759	1.021968
	XgD	0.178251	1.022753	1.017489	1.018073	1.022919
	AkD	0.201712	1.046044	1.044679	1.044851	1.044983

SELF are reported in Table 14. Besides, the confidence limits of BCIs using different classical methods of estimation are reported in Table 13. From Table 13, it was found that for data set I MLE and LnD give the best performance as compared to other methods and distributions, respectively. Similarly, for data set II, MPSE and XgD play the same role. In the different BCIs, $\mathcal{STB}$ for data set I and $\mathcal{BCPB}$ for data set II perform better. It is observed that the width of the HPD is the minimum among the widths of BCIs, which shows similar trends of inference as seen in the simulation study. Specifically, LnD gives the least HPD for Data Set I and XgD gives the least HPD for Data Set II. From Tables 12 and 14, we observe that the estimated value of $\mathcal{C}_{py}$ (under LnD and AkD) based on different methods of estimation indicates that the process is almost capable, i.e., the process is satisfactory from a capability point of view even though it is under statistical control.

Table 13 Widths of BCIs for $\mathcal{C}_{py}$ under different method of estimation for different models

Est.	Data set - I				Data set - II			
	Widths of $\mathcal{C}_{py}$ for LnD							
	$\mathcal{SB}$	$\mathcal{PB}$	$\mathcal{STB}$	$\mathcal{BCPB}$	$\mathcal{SB}$	$\mathcal{PB}$	$\mathcal{STB}$	$\mathcal{BCPB}$
MLE	0.007125	0.006571	0.000601	0.003240	0.025848	0.025948	0.000909	0.006820
LSE	0.007396	0.006559	0.000517	0.002880	0.023367	0.020624	0.000472	0.002329
WLSE	0.006622	0.005997	0.000268	0.001548	0.022409	0.019861	0.000240	0.001610
MPSE	0.006426	0.005871	0.000612	0.000599	0.000899	0.017821	0.000215	0.000158
	Widths of $\mathcal{C}_{py}$ for XgD							
	$\mathcal{SB}$	$\mathcal{PB}$	$\mathcal{STB}$	$\mathcal{BCPB}$	$\mathcal{SB}$	$\mathcal{PB}$	$\mathcal{STB}$	$\mathcal{BCPB}$
MLE	0.013944	0.013003	0.007881	0.012974	0.015982	0.014896	0.000433	0.002391
LSE	0.017734	0.017636	0.011626	0.018388	0.032925	0.029829	0.010935	0.031626
WLSE	0.015281	0.014583	0.009184	0.014583	0.029070	0.025405	0.009756	0.026621
MPSE	0.014752	0.013251	0.008131	0.013258	0.024657	0.023337	0.000401	0.000388
	Widths of $\mathcal{C}_{py}$ for AkD							
	$\mathcal{SB}$	$\mathcal{PB}$	$\mathcal{STB}$	$\mathcal{BCPB}$	$\mathcal{SB}$	$\mathcal{PB}$	$\mathcal{STB}$	$\mathcal{BCPB}$
MLE	0.035285	0.035945	0.030909	0.037434	0.078136	0.080280	0.050420	0.071256
LSE	0.041131	0.040834	0.028299	0.042009	0.071672	0.072382	0.049761	0.073138
WLSE	0.043836	0.044732	0.035815	0.042761	0.099522	0.100507	0.066170	0.100507
MPSE	0.038869	0.039531	0.030182	0.039732	0.057403	0.056532	0.049625	0.083674

Table 14 Bayes estimates of Cpy through M-H algorithm with corresponding risk and HPD credible intervals

Model	Data set - I			Data set - II		
	Bayes estimate and HPD interval					
	Bayes est	risk	HPD	Bayes est	risk	HPD
LnD	1.000192	0.000002	0.000402	1.018533	0.000080	0.000289
XgD	0.990712	0.000020	0.001643	1.019016	0.000020	0.000131
AkD	1.036033	0.000002	0.001532	1.041075	0.000058	0.000550

7 Conclusions

In this research, we looked at four traditional methods of GPCI $\mathcal{C}_{py}$ point estimate (MLE, LSE, WLSE, and MPSE) as well as the Bayesian method (M-H algorithm) and demonstrated the proposed methods with two real-life instances. We conducted simulation research to compare these strategies with different sample sizes and different combinations of the unknown parameters because it is not possible to compare these methods conceptually. For the GPCI $\mathcal{C}_{py}$, we examined BCIs and HPD intervals in addition to point estimation.

Simulation study results show that the performance of the M-H algorithm is satisfactory. Further, simulation results suggest that for almost all the cases, Bayes estimates perform better than classical methods of estimation. It's worth noting that the prior distributions' hyper-parameters must be carefully chosen. Among the other conventional methods of estimation, MPSE produces the best results in terms of MSEs for practically all sample sizes and parameter values. Among the considered BCIs, $\mathcal{STB}$ performed better in terms of $\mathcal{AW}s$. Also, the $\mathcal{AW}s$ of HPD under SELF are smaller than considered BCIs. The data analysis also echoed the similar pattern of results that we have observed in the simulation study. As a result of the entire analysis, we can conclude that LnD outperforms XgD and AkD for almost all paremeter values except $\psi = 1.25$, and that the performance level of the investigated distribution is $LnD > XgD > AkD$. I believe that if this research approach works well, the industries will be able to use it in the future to evaluate the capabilities of any process distribution.

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