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QUEUING SYSTEM WITH INVERSE GAUSSIAN AND MUKHERJEE-ISLAM SERVICE TIME DISTRIBUTIONS

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Abstract

A single server waiting line system with Poisson arrivals and Inverse Gaussian service times has been analyzed. The same model has been analyzed by using Mukherjee-Islam (1983) model as the service time model. Inverse Gaussian model has been used in queuing theory but the Mukherjee-Islam model which is basically a failure model has been tested first time as the service time model.

Keywords: Queuing model, service time distribution, arrival rate, system

1. Introduction

The Inverse Gaussian family of distributions are often used in analyzing many of the realistic situations arising at life testing, economical analysis, insurance studies etc. The major advantage of this distribution is the interpretation of the Inverse Gaussian random variable as the first passage time distribution of Brownian motion with positive drift. In textile industries the printing or bleaching processed are distributed approximately as Inverse Gaussian distribution. Here unit of cloth is to be taken as customer, the printing or bleaching is viewed as service. In spite of wide applicability of the Inverse Gaussian distribution as approximate model for skewed data and having simple exact sampling theory, it has not been used much in analyzing waiting line systems.

2. Queuing Model with Inverse Gaussian Service Time Distribution

Consider a single server queuing with infinite capacity having FCFS (First Come First Serve) queue discipline and the arrivals are Poisson with arrival rate λ. The service time distribution of the process is an Inverse Gaussian of the form;

$$f(t;\mu,\theta) = \left(\frac{\theta}{2\pi t^3}\right)^{1/2} \exp-\left\{\frac{\theta(\mu t-1)^2}{2.t}\right\} \; ; \quad \mu,\ \theta \geq 0 \; ; \quad 0 \leq t < \infty \qquad (1)$$

with Mean = $1/\mu$ and Variance = $1/\theta\mu^3$.

2.1 Maximum Likelihood Estimates

Parameters μ and θ are involved in the service time distribution given in equation (1). Consider a random sample t1,t2,…..,tn from the population with p.d.f. (1). The likelihood function is given as;

$$L(t\ ;\mu,\theta) = \prod_{i=1}^{n} f_i(t_i;\mu,\theta)$$

$$= \prod_{i=1}^{n}\left(\frac{\theta}{2\pi t_i^3}\right)^{1/2} \exp-\left\{\frac{\theta(\mu t_i-1)^2}{2.t_i}\right\}$$

$$= \left(\frac{\theta}{2\pi}\right)^{n/2} \prod_{i=1}^{n}\left(\frac{1}{t_i}\right)^{3/2} \exp-\left\{\frac{\theta}{2}\sum_{i=1}^{n}\frac{(\mu t_i-1)^2}{t_i}\right\} \qquad (2)$$

which gives the M.L.E.'s for μ and θ as ;

$$\hat{\mu} = \frac{n}{\sum_{i=1}^{n} t_i} = \frac{1}{\bar{T}} \qquad (3)$$

and

$$\hat{\theta} = \frac{n}{\sum_{i=1}^{n}\frac{(\mu t_i-1)^2}{t_i}}$$

$$= \frac{n.\bar{T}^2}{\sum_{i=1}^{n}\frac{(t_i-\bar{T})^2}{t_i}} \qquad (4)$$

2.2 Analysis of the Model

Let Hn be the probability that there are n arrivals during the service time of a customer. Let H(z) denotes the probability generating function (p.g.f.) of Hn given as

$$H(z) = \sum_{n=1}^{\infty} H_n z^n \ ; \ |z| \le 1 \qquad (5)$$

Following heuristic argument of Kendall (1953) and Gross and Hariss (1974), the probability Hn that there are n arrivals during the service time is given by,

$$H = \int_0^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!}\left(\frac{\theta}{2\pi t^3}\right)^{1/2} \exp-\left\{\frac{\theta(\mu t-1)^2}{2.t}\right\} dt \qquad (6)$$

Then the probability generating function of Hn is

$$H(z) = \sum_{n=0}^{\infty} z^n \int_0^{\infty} \frac{e^{-\lambda t} (\lambda t)^n}{n!} \left(\frac{\theta}{2\pi t^3}\right)^{1/2} \exp - \left\{\frac{\theta(\mu t-1)^2}{2.t}\right\} dt$$

which, finally comes out to be,

$$H(z) = \exp\left\{\theta\mu\left[1-\left(1+\frac{2(\lambda-\lambda z)}{\theta\mu^2}\right)^{1/2}\right]\right\} \quad (7)$$

The average number of arrivals during the service time is

$$H'(z)\Big|_{z=1} = \frac{\lambda}{\mu} \quad (8)$$

If Pn is the probability that there are n customers in the system at the steady state and P (z) the probability generating function of Pn, then,

$$P(z) = \frac{\left(1-\frac{\lambda}{\mu}\right)(1-z)\exp\left\{\theta\mu\left[1-\left(1+\frac{2(\lambda-\lambda z)}{\theta\mu^2}\right)^{1/2}\right]\right\}}{\exp\left\{\theta\mu\left[1-\left(1+\frac{2(\lambda-\lambda z)}{\theta\mu^2}\right)^{1/2}\right]\right\}-z} \quad (9)$$

By expanding P (z) and collecting the coefficients of zn, we get Pn the probability that there are n customers in the system.

The probability that the system is empty is

$$P_0 = 1-\frac{\lambda}{\mu} \quad (10)$$

After substituting estimated value of μ in equation (10), P0 can be evaluated for different values of λ. It is also observed that P_0 is independent of θ i.e. P0 is not influenced by the variability of the service time.

The average number of customers in the system is obtained as ,

$$L = P'(z)\Big|_{z=1}$$

$$= \frac{\lambda\left[\lambda+\theta\mu\left(2\mu-\lambda\right)\right]}{2\left(\mu-\lambda\right)\theta\mu^2} \quad (11)$$

From the equation (11) it can be observed that the average number of customers in the system is influenced by θ. The value of L can be computed by using estimated values of μ and θ and different values of λ.

The variability of the system size can be obtained by using the formula.

$$V = [P''(z) + P'(z) - (p'(z))^2]\Big|_{z=1}$$

$$= \frac{3A^2 + (\rho-1)\{2\rho-3).A - 4.B - 6\rho(2\rho-1)(\rho-1)}{12(\rho-1)^2} \quad (12)$$

where

$$A = \frac{\lambda^2}{\mu^2}\left(\frac{1}{\theta\mu}+1\right)$$

$$B = \frac{\lambda^3}{\mu^3}\left(\frac{3}{\theta\mu^2}+\frac{3}{\theta\mu}+1\right)$$

and $$\rho = \frac{\lambda}{\mu}$$

The coefficient of variation (C.V.) of the system size will be

$$C.V. = \frac{\sqrt{V}}{L} x100 \qquad (13)$$

For the different values of λ and estimated values of μ and θ, the values of variability of the system size and the coefficient of variation can be computed.

It has been studied by Murty (1993) that the variability of the system size decreases as μ increases for fixed values of λ and θ. As λ increases the variability of the system size increases for fixed values of θ and μ. Further it has been observed that the coefficient of variation increases as μ increases for fixed values of 'λ' and 'θ'. As λ increases the coefficient of variation decreases for fixed values of θ and λ. As θ increases the coefficient of variation decreases for fixed values of 'λ' and 'θ'. For given arrival and service rates, the mean queue length of M/M/1 and M/1G/1 model are compared and it has been observed that when $\frac{1}{\mu} \leq \theta,$ the mean queue length of M/IG/1 is less than that of the M/M/1 model.

It concludes that by controlling 'θ', the mean queue length of M/IG/1 model can be controlled, which has influence on the optimal operating policies of the system. Further the model can be analyzed in a better way by using estimated values of θ and μ in place of hypothetical values of θ and μ. In this model we have used only hypothetical values of λ.

3. Queuing Model with Mukherjee-Islam Service Time Distribution

Again consider a single server queuing with infinite capacity having FCFS (First Come First Serve) queue discipline. The arrivals are assumed to be Poisson with arrival rate λ. But the service time distribution of the process is a new finite range probability distribution which was originally introduced by Mukherjee-Islam (1983) as a life testing model.

$$f(t;\, \theta,p) = (p/\theta^p)\, t^{p-1}; \quad p,\, \theta > 0; \quad t \geq 0 \qquad (14)$$

The above model is monotonically decreasing and highly skewed to the right. The graph is J-shaped thereby showing the uni-modal feature. The distribution function of above model is;

$$F(t) = [t/\theta]^p \qquad (15)$$

with Mean = $\frac{p}{p+1}.\theta$

and Variance = $\frac{p}{(p+1)^2(p+2)}.\theta^2$

3.1 Maximum Likelihood Estimates

For the sample of size n, the likelihood function for the model will be given by

$$L(t;\ \theta,\ p) = p^n\theta^{-np}\prod_{i=1}^{n} t_i^{p-1} \tag{16}$$

The M.L.E.'s of p and θ obtained from (16) are

$$\hat{p} = \frac{n}{n\log\theta - \sum \log t_i} \tag{17}$$

$$\hat{\theta} = t_{(n)} = \text{Max}\ (t_1, t_2, \ldots\ldots., t_n) \tag{18}$$

3.2 Analysis of the model

Let Hn be the probability that there are n arrivals during the service time of a customer. Let H(z) denotes the probability generating function (p.g.f.) of Hn given as

$$H(z) = \sum_{n=1}^{\infty} H_n z^n\ ; |z| \le 1$$

Following heuristic argument of Kendall (1953) and Gross and Hariss (1974), the probability Hn that there are n arrivals during the service time is given by

$$H_n = \int_0^{\theta} \frac{e^{-\lambda t}(\lambda t)^n}{n!}\left(\frac{p}{\theta^p}\right)t^{p-1}\,dt \tag{19}$$

Then the probability generating function of Hn is

$$H(z) = \sum_{n=0}^{\infty} z^n \int_0^{\theta} \frac{e^{-\lambda t}(\lambda t)^n}{n!}\left(\frac{p}{\theta^p}\right)t^{p-1}\,dt$$

which gives

$$H(z) = p.\sum_{j=0}^{\infty} \frac{(-(\lambda-\lambda z))^j}{j!}\frac{\theta^j}{p+j} \tag{20}$$

The average number of arrivals during the service time is

$$H'(z)\Big|_{z=1} = \frac{p}{p+1}.\theta.\lambda \tag{21}$$

Let $\mu = \frac{p+1}{p}.\theta$ (the reciprocal of the mean) then

$$H'(z)\Big|_{z=1} = \frac{\lambda}{\mu} \tag{22}$$

Now, let P_n be the probability that there are 'n' customers in the system at the steady state and P(z) be the probability generating function of P_n. Then by expanding P(z) and collecting the coefficients of z^n, we get P_n.

Furthermore, the analysis can be carried out in the same manner as in previous section for Inverse Gaussian service time distribution system. The successful application of these models as service time models in queuing theory leads the enlargement of the family of such distributions.

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