Fractals as Julia and Mandelbrot Sets via S-iteration

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Abstract

To understand the phenomena of expanding symmetries Fractals patterns are an important tool which exhibit similar patterns for different scales. In the present paper, establishing an escape criteria by using S-iteration process to visualize fractals namely Julia and Mandelbrot sets for the function $F(w) = ae^{w^p} + c$ where $c, a \in \mathbb{C}$ and $p \geq 2$. The result obtain is a generalization of the existing algorithm and technique providing fractals for different parameter values. Also, the time taken to obtain fractals for different parameters by using computer software MATLAB is computed in seconds.

Keywords: Fractals, Julia set, mandelbrot set, escape criteria.

1 Introduction

Fractal patterns are form by replicating its shape at finer scales. To understanding natural or living phenomena like microorganisms fractals are an important tool [1]. Fractals are also commonly found in nature like crystals, rivers, branches of trees, coast lines, electricity, clouds and so on [2]. Radio

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signal and wavelength in wireless communication sector is an application of fractals [4]. Fractals are useful in fluid dynamics and mechanics to understand the behaviour of the streams. In addition, cryptography, encryption and image compression are also applications of the fractals [5]. The collection of points where complex valued functions shows chaotic behaviour is called Julia set and their collection is called Mandelbrot set. Similarity in a fractal at every point of is the same as the entire, for which any appropriate focussed part is more subdued when increased or decreased. In 1978 Benoit Mandelbrot used the term fractal [2], and from thereon he is known as the "father of fractal geometry". For the generation of fractals, the role of the iterative scheme of fixed point theory is very important. Recently lots of authors used different iterative processes to generate the fractals for functions like sine,cosine,complex,exponential and so on [7]–[19].

In the present work, we use S-iterative process [20], to establish the escape criteria for visualizing fractals as Julia set and Mandelbrot set for function $F(w) = ae^{w^p} + c$ where $c, a \in \mathbb{C}$ and $p \geq 2$, we have given several examples to observe changes in graphical images and analyse the influence of underlying parameters on the dynamics, generation time, colour and shape variations of produced fractals. The coloured points within the resulting fractals represent "escape points," meaning they approach infinity through the iteration method. The diverse colours indicate the rate at which a point escapes.

2 Preliminaries

We start with fundamental notions and definitions which are essential for this research.

Definition 2.1 (Julia set [3, 21]) *The collection of points within a set of complex numbers so that the trajectory of the function $F : \mathbb{C} \rightarrow \mathbb{C}$,diverging towards infinity, is termed as filled Julia set of F.We can write*

$$S_F = \{w \in \mathbb{C} : \{|F^i(w)|\}_{i=0}^{\infty} \text{ is bounded}\} \tag{1}$$

The Julia set of F is the boundary of S_F.

Definition 2.2 (Mandelbrot set [22, 23]) *The collection of parameters within a set of complex numbers so that the filled Julia set S_F of $F = w^2+c$ is connected is termed as Mandelbrot set, Mathematically it is defined as :*

$$M = \{c \in \mathbb{C} : S_F \text{ is connected}\}.$$

The Mandelbrot set M encapsulates significant information related to the Julia set and can also be expressed as:

$$M = \{c \in \mathbb{C} : |F(w)| \nrightarrow \infty \, as \, k \to \infty\}. \tag{2}$$

Definition 2.3 (S-iteration [20]) *For $w_0 \in \mathbb{C}$, S-iteration is defined as*

$$\begin{cases} w_{i+1} = (1-\alpha_1)F(w_i) + \alpha_1 F(z_i), \\ z_i = (1-\alpha_2)w_i + \alpha_2 F(w_i), \end{cases}, \tag{3}$$

where $\{\alpha_1\}$,$\{\alpha_2\}$ $\in$ [0,1] and i = 0,1,2. . . .

3 Main Results

To generate complex fractals escape criteria play an important role. We present the escape criterion for the function via S-iteration[[20]]. S-iteration have two steps. The dependency on w_i extends to all z and w within the complex number set $\mathbb{C}$. Specifically, for $i = 0$, in this article, we make the assumption that $w_0 = w$ and $z_0 = z$.

3.1 Escape Criteria of S-iteration

Suppose $F(w) = ae^{w^p} + c$ where $c, a \in \mathbb{C}$ and $p \geq 2$ be a function, then

$$\begin{aligned} |e^{w^p}| &= \left|1 + w^p + \frac{w^{2p}}{2!} + \frac{w^{3p}}{3!} + \frac{w^{4p}}{4!} \dots\right| \\ &> |w^p| \left|1 + \frac{w^p}{2!} + \frac{w^{2p}}{3!} + \frac{w^{3p}}{4!} \dots\right| \\ &> |w^p||m_1|, \end{aligned} \tag{4}$$

where $|m_1| \in (0, 1]$ satisfying the bound $|m_1| < \left|1 + \frac{w^p}{2!} + \frac{w^{2p}}{3!} + \frac{w^{3p}}{4!} \dots\right|$; $w \in \mathbb{C}$, and similarly,

$$\begin{aligned} |e^{z^p}| &= \left|1 + z^p + \frac{z^{2p}}{2!} + \frac{z^{3p}}{3!} + \frac{z^{4p}}{4!} \dots\right| \\ &> |z^p| \left|1 + \frac{z^p}{2!} + \frac{z^{2p}}{3!} + \frac{z^{3p}}{4!} \dots\right| \\ &> |z^p||m_2|, \end{aligned} \tag{5}$$

where $|m_2| \in (0,1]$ satisfying the bound $|m_2| < \left|1 + \frac{z^p}{2!} + \frac{z^{2p}}{3!} + \frac{z^{3p}}{4!} \dots\right|$; $z \in \mathbb{C}$.

Theorem 3.1 *Let* $F(w) = ae^{w^p} + c$ *where* $p \geq 2$, c, $a \in \mathbb{C}$ *and* $\{w_i\}_{i\in\mathbb{W}}$ *be the S-iteration defined in* (3) *with* $|w| \geq |c| > \left(\frac{2}{\alpha_2|a||m_1|}\right)^{\frac{1}{(p-1)}}$ *and* $|w| \geq |c| > \left(\frac{2}{|a|(\alpha_1|m_2|-|m_1|)}\right)^{\frac{1}{(p-1)}}$. *Then* $|w_i| \to \infty$, *as* $i \to \infty$.

Proof. Firstly we can consider

$$|z_i| = |(1-\alpha_2)w + \alpha_2 F(w)|$$

Then for i= 0 and from $F(w) = ae^{w^p} + c$ we get,

$$|z| = |(1-\alpha_2)w + \alpha_2(ae^{w^p} + c)|$$

Next, by using (4), and the given fact $|w| \geq |c| > \left(\frac{2}{\alpha_2|a||m_1|}\right)^{\frac{1}{(p-1)}} \Rightarrow$ $(\alpha_2|a||m_1||w^{p-1}| - 1) > 1$,
we have,

$$\begin{aligned}
|z| &= |(1-\alpha_2)w + \alpha_2(ae^{w^p} + c)| \\
&\geq |\alpha_2 ae^{w^p} + (1-\alpha_2)w| - |\alpha_2 c| \\
&\geq |\alpha_2 ae^{w^p}| - |(1-\alpha_2)w| - |\alpha_2 w|, \because |w| \geq |c| \\
&\geq |\alpha_2 am_1 w^p - |w| + |\alpha_2 w| - |\alpha_2 w|, (From (4)) \\
&\geq \alpha_2|a||m_1||w^p| - |w| \\
&\geq |w|(\alpha_2|a||m_1||w^{p-1}| - 1) \\
&\geq |w|.
\end{aligned}$$

Next, in the second step of S-iteration,

$$|w_{i+1}| = |(1-\alpha_1)F(w) + \alpha_1 F(z)|.$$

for i=0 we have from (6), and $|z| \geq |w| \geq |c| > \left(\frac{2}{|a|(\alpha_1|m_2|-|m_1|)}\right)^{\frac{1}{(p-1)}}$ $\implies (|a|(\alpha_1|m_2| - |m_1|))|w^{p-1}| - 1) > 1$,

$$\begin{aligned}
|w_1| &= |(1-\alpha_1)(ae^{w^p} + c) + \alpha_1(ae^{z^p} + c)| \\
&\geq |\alpha_1(ae^{z^p} + c)| - |(1-\alpha_1)(ae^{w^p} + c)|
\end{aligned}$$

$$\geq |\alpha_1 ae^{z^p}| - |\alpha_1 c| - |ae^{w^p}| - |c| + |\alpha_1 ae^{w^p}| + |\alpha_1 c|$$
$$\geq |\alpha_1 ae^{z^p}| - |ae^{w^p}| + |\alpha_1 ae^{w^p}| - |c|$$
$$\geq |\alpha_1 ae^{z^p}| - |ae^{w^p}| - |c|, (Neglecting\ |\alpha_1 ae^{w^p}|)$$
$$\geq \beta|m_3||x^p| - |x|$$
$$\geq |\alpha_1 am_2 z^p| - |am_1 w^p| - |c|, (From(4)and(5))$$
$$\geq |\alpha_1 am_2 w^p| - |am_1 w^p| - |w|, (\because |z| \geq |w| \geq |c|)$$
$$\geq |w|(\alpha_1|a||m_2||w^{p-1}| - |a||m_1||w^{p-1}| - 1)$$
$$|w_1| \geq |w|((|a|(\alpha_1|m_2| - |m_1|))|w^{p-1}| - 1)$$

Following the same pattern, for i=1,

$$|w_2| \geq |w|((|a|(\alpha_1|m_2| - |m_1|))|w^{p-1}| - 1)^2.$$

Next, for i=2, we have

$$|w_3| \geq |w|((|a|(\alpha_1|m_2| - |m_1|))|w^{p-1}| - 1)^3.$$

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$$|w_{i+1}| \geq |w|((|a|(\alpha_1|m_2| - |m_1|))|w^{p-1}| - 1)^i.$$

Since $|w| \geq |c| > \left(\frac{2}{|a|(\alpha_1|m_2|-|m_1|)}\right)^{\frac{1}{(p-1)}} \implies ((|a|(\alpha_1|m_2| - |m_1|))|w^{p-1}| - 1) > 1$ and therefore, $|w_i| \to \infty$ as $i \to \infty$.

Corollary 3.1 *For $k \geq 0$, if*

$$\left\{|w_i| > w_0 > max\left\{|c|, \left(\frac{2}{\alpha_2|a||m_1|}\right)^{\frac{1}{(p-1)}}, \left(\frac{2}{|a|(\alpha_1|m_2|-|m_1|)}\right)^{\frac{1}{(p-1)}}\right\}\right\},$$

then there exists a positive number $\theta > 0$ so that

$$|w|\Big((\alpha_2|a||m_1|)(|a|(\alpha_1|m_2| - |m_1|))|w^{p-1}| - 1\Big) > 1 + \theta$$
$$\implies |w_{k+i}| > |w_k|(1+\theta)^{k+i}$$

and then $|w_i| \to \infty$ as $i \to \infty$.

4 Visualization of Fractals for the Function $F(w) = ae^{w^p} + c$

To visualize the Julia sets we use Algorithm 1 and for Mandelbrot sets, Algorithm 2 , for obtaining fractals for logarithmic function using S-iteration scheme via MATLAB (R2015a) and colourmap (Figure 1). In this iterative process, numerous fractals emerge as Julia and Mandelbrot sets. It is noteworthy that many of these fractals exhibit symmetry. This resemblance proves advantageous in both the Textile Industry and interior decoration.

4.1 Julia Set

We visualize some Julia sets of function $F(w) = ae^{w^p} + c$ where $c, a \in \mathbb{C}$ and $p \geq 2$ for different values of parameters using S-iteration. We have considered the maximum number of iterations are 50 (i.e., P=50).

Figure 1 Colourmap for the visualization of fractals.

Algorithm 1: For visualization of Julia Set

Input: $F(w) = ae^{w^p} + c$ where $c, a \in \mathbb{C}$ and $p \geq 2$; $p \geq 2$,$A \subset \mathbb{C}$ is area;
P is maximum number of iterations;$\alpha_1, \alpha_2 \in (0, 1]$-parameter of S iteration.
Colourmap [0..C-1]-colour with C colours.
Output: Julia set for area A.
for $w_0 \in A$ **do**
R=Stoping threshold for S iteration
i=0
while $i \leq P$ **do**
$w_{i+1} = (1 - \alpha_1)F(w_i) + \alpha_1 F(z_i)$
$z_i = (1 - \alpha_2)w_i + \alpha_2 F(w_i)$
if $|w_{i+1}| > R$ **then**
break
end if
i=i+1
$j = [(C - 1)\frac{i}{P}]$
colour w_0 with colourmap [i]
end for

Table 1 Parameter values for Figure 2

Sr.No.	**c**	**a**	α_1	α_2	m_1	m_2	**p**
(a)	0.01i	1	0.2	0.5	0.07	0.4	2
(b)	0.01i	1	0.2	0.5	0.7	0.4	2
(c)	0.01	i	0.05	0.02	0.07	0.03	4
(d)	0.01	i	0.05	0.02	0.07	0.03	5
(e)	0.01	1	0.2	0.5	0.07	0.4	6
(f)	0.01	0.65	0.2	0.5	0.07	0.4	7

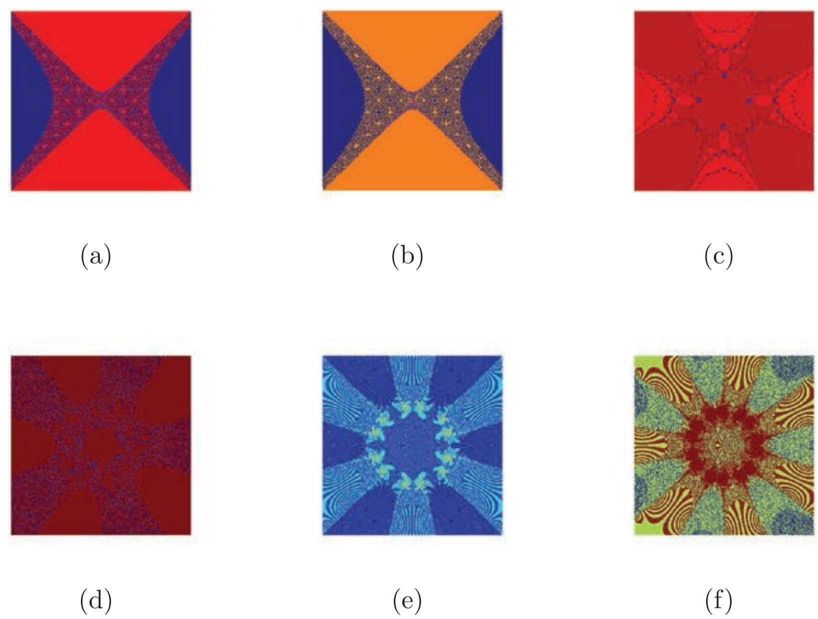

Figure 2 For the different values of c and p. **(a)** A = [-50,50]×[-50,50] and time 2.287195 s. **(b)** A =[-50,50]×[-50,50] and time 2.314692 s. **(c)** A = [-2,2]×[-2,2] and time 2.412298 s. **(d)** A = [-2,2]×[-2,2] and time 2.827401 s. **(e)** A = [-2,2]×[-2,2] and time 2.790364 s. **(f)** A = [-2,2]×[-2,2] and time 2.917099 s.(Where A = Area).

We observe that Julia fractals appear like the traditional Kachhi Thread Works found in the Kutch district of Gujarat, India, and the number of outer bulbs is equal to the value of p. Surprisingly, the color for different values of p is different.(See Table 1 and Figure 2).

Table 2 Parameter values for Figure 3

Sr.No.	**c**	**a**	α_1	α_2	m_1	m_2	**p**
(a)	i	1	0.02	0.05	0.07	0.04	12
(b)	i	1	0.02	0.05	0.07	0.04	25
(c)	i	1	0.02	0.05	0.07	0.04	30
(d)	i	1	0.02	0.05	0.07	0.04	35
(e)	i	1	0.02	0.05	0.07	0.04	100
(f)	1	1	0.02	0.05	0.07	0.04	200

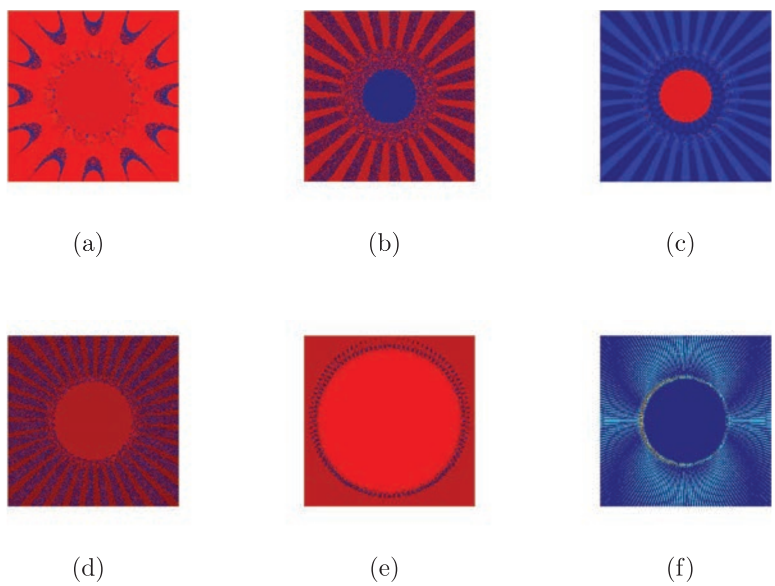

Figure 3 For different values of p. **(a)** A = [-2,2]×[-2,2] and time 2.942379 s. **(b)** A = [-2,2]×[-2,2] and time 4.869238 s. **(c)** A = [-2,2]×[-2,2] and time 5.935421 s. **(d)** A = [-2,2]×[-2,2] and time 4.260984 s. **(e)** A = [-1.2,1.2]×[-1.2,1.2] and time 5.107311 s. **(f)** A = [-2,2]×[-2,2] and time 3.909764 s.

Higher order Julia set for fixed values of parameters is obtained. Julia fractals approach circular shapes as the value of p increases (See, Table 2 and Figure 3). These fractals share striking similarities with sunflower..

4.2 Mandelbrot set

Here we discuss several Mandelbrot sets for the function $F(w) = ae^{w^p} + c$ where $c, a \in \mathbb{C}$ and $p \geq 2$ for different values of p in the trajectory of

S-iteration. We have generated Mandelbrot sets for various parameters via S-iteration. For consistency across all fractals, we have set the maximum iteration to 50 (i.e., P=50) in Algorithm number 2.

Algorithm 2: For visualization of Mandelbrot Set

Input: $F(w) = ae^{w^p} + c$ where $c, a \in \mathbb{C}$ and $p \geq 2$; $p \geq 2$,$A \subset \mathbb{C}$-area; P-maximum number of iterations;$\alpha_1, \alpha_2 \in (0, 1]$-parameter of S iteration. Colourmap [0..C-1]-colour with C colours.
Output: Mandelbrot set for area A.
for $w_0 \in A$ **do**
R=Stoping threshold for S iteration
i=0
while $i \leq P$ **do**
$w_{i+1} = (1 - \alpha_1)F(w_i) + \alpha_1 F(z_i)$
$z_i = (1 - \alpha_2)w_i + \alpha_2 F(w_i)$
if $|w_{i+1}| > R$ **then**
break
end if
i=i+1
$j = [(C - 1)\frac{i}{P}]$
colour w_0 with colourmap [i]
end for

Table 3 Parameter values for Figure 4

Sr.No.	a	α_1	α_2	m_1	m_2	p
(a)	0.1	0.2	0.5	0.1	0.3	2
(b)	0.1	0.2	0.5	0.1	0.3	3
(c)	1	0.02	0.05	0.01	0.03	4
(d)	2	0.2	0.5	0.5	0.9	3
(e)	2	0.2	0.5	0.5	0.9	4
(f)	2	0.2	0.5	0.5	0.9	5

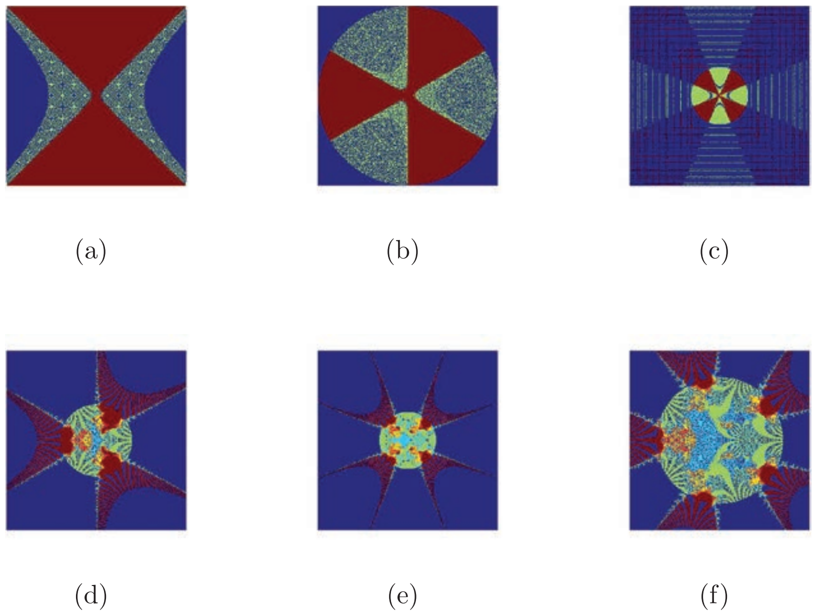

Figure 4 For different values of p effects on the Mandelbrot set in S-iteration. **(a)** A = [-50,50]×[-50,50] and time 3.973087 s. **(b)** A = [-20,20]×[-20,20] and time 3.978889 s. **(c)** A = [-50,50]×[-50,50] and time 4.547364 s. **(d)** A = [-5,5]×[-5,5] and time 4.086939 s. **(e)** A = [-5,5]×[-5,5] and time 4.732100 s. **(f)** A = [-2,2]×[-2,2] and time 4.930074 s.

Mandelbrot fractals appear as in Figure 4, and the number of outer bulbs is equal to the numerical values of p for entries in Table 3.

Table 4 Table 4: Parameter values for Figure 5

Sr.No.	**a**	α_1	α_2	m_1	m_2	**p**
(a)	0.1	0.001	0.009	0.1	0.9	10
(b)	0.1	0.001	0.009	0.1	0.9	20
(c)	0.1	0.001	0.009	0.1	0.9	50
(d)	0.5	0.001	0.009	0.1	0.9	50
(e)	1	0.001	0.009	0.1	0.9	50
(f)	2	0.001	0.009	0.1	0.9	50

Higher order Mandelbrot set is obtained. As the value of p increases, Mandelbrot fractals approach circular shapes. Refer to Figure 5 and Table 4 for details. As value of a increase the changes in fractal shape can be seen in Figure 5(d)–5(f).

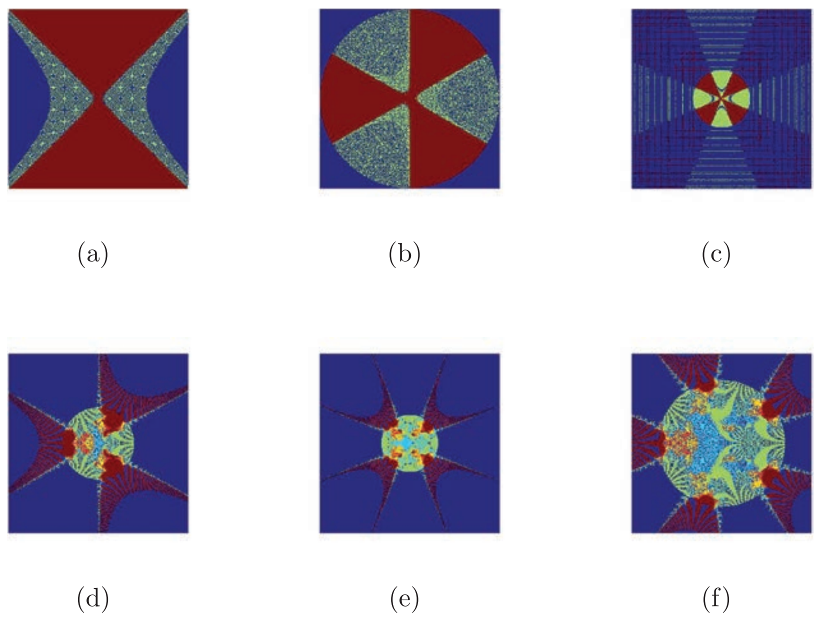

Figure 5 For different values of a and p effect on Mandelbrot set in S-iteration. **(a)** A = [-3,3]×[-3,3] and time 4.467746 s. **(b)** A = [-2,2]×[-2,2] and time 4.548568 s. **(c)** A = [-1.2,1.2]×[-1.2,1.2] and time 5.420616 s.**(d)** A = [-1.2,1.2]×[-1.2,1.2] and time 5.795210 s. **(e)** A = [-1.2,1.2]×[-1.2,1.2] and time 5.967584 s. **(f)** A = [-1.2,1.2]×[-1.2,1.2] and time 6.157547 s.

5 Conclusions

The escape criteria for the function $F(w) = ae^{w^p} + c$, where $c, a \in \mathbb{C}$ and $p \geq 2$, have been shown using S-iteration. By applying the outcomes of Algorithms 1 and 2 in S-orbit, the Julia and Mandelbrot sets have been visualised. With thorough explanations, we have talked about the created Julia sets and Mandelbrot sets for various values. We also noticed that the number of repellers and attractor is different for different values of p. The fractals also get more colourful for complex values of c, and the mandelbrot set starts to take on a circular shape for higher values of p. Additionally, we computed the execution time in seconds for the fractal generation, demonstrating that the execution time varies for various parameters in the generations of fractals as defined by the Julia and Mandelbrot sets. We have looked at many approaches to creating images and assessed how parameters affect the dynamics, colour, and general look of fractals. Some of the fractals in our collection have an amazing resemblance to the traditional Kachhi Thread Works from Gujarat's

Kutch district and to Rangoli patterns created throughout India. There are several benefits to this similarity for the interior design and textile industries. Furthermore, some of the fractals in our collection exhibit a remarkable similarity to the spinning wheel that is traditionally used on Diwali. These results can be applied to determine the usage of various iterative methods in fractal formation.

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