Jarlskog Determinant in Four Flavor Neutrino Oscillation Framework

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Received 20 January 2022; Accepted 11 February 2022; Publication 07 May 2022

Abstract

The weak CP voilation in leptonic sector is potentially one of the key aspect to probe the physics beyond the Standard Model (SM). This article majorly focus on the Dirac CP phase effect on the Jarlskog Deteminant in four flavor neutrino framework. The calculations for the upper and lower bound of the Jarlskog Determinant in 4×4 neutrino four flavor mixing matrix has also been performed.

Keywords: CP violaton, Jarlskog determinant, PMNS Matrix.

Journal of Graphic Era University, Vol. 10_2, 83–94.
doi: 10.13052/jgeu0975-1416.1022

1 Introduction

The experimental detection of neutrino oscillation came from Ray Davis's Homestake experiment and Y. Fukuda et al. [1]. Several other collaborations have neutrino oscillation data obtained from the teresstrial and non-teresstial experiments. In attempt to interpret the neutrino oscillation data obtained from the atmospheric, solar and LSND collabration experiments we have to incorporate the four neutrino of definite mass.

The three different scale of neutrino mass-squared differences $\triangle m^2_{sun} \ll \triangle m^2_{atm} \ll \triangle m^2_{LSND}$ obtained by different neutrino oscillation experiments and LSND collabration [2]. It indicates the possibility of four neutrinos with definite mass to explain these data. Super- Kamiokande Collaboration, Y. Fukuda et al. [1] experimental data was in good agreement with two flavor oscillation i.e.$\nu_\mu \rightarrow \nu_\epsilon$ and provided evidence for neutrino oscillation. All these neutrino oscillation data could be consolidated by four neutrinos with definite mass by S.M. Bilenky et al. [3]. Thus four-neutrino models have been studied by earlier authors [15, 20, 21, 23]. The simultaneity in the oscillations of four different flavors of neutrinos demands an inclusion of a light sterile neutrino in the interpretation. The four neutrino mixing model are expected to violate the CP and T symmetry, the measurement of the violating factors require new generation very long baseline neutrino experiments. Some efforts are made in this directions suggest the possibilities to observe four neutrino oscillation. The better understanding of the nontrivial complex phases observed in 4×4 lepton flavor mixing matrix can be achieved by investigating the specific parameters involved between CP violating phases and flavor 4×4 lepton flavor mixing angle with rephrasing invariants of mixing matrix. Some attempts were made, but analytically correct result could not be achieved. The presence of $1-\gamma_5$ factors in Lagrangian indicates, parity violation is maximal in charged-current interactions. Similarly charge-conjugation invariance is maximally violated in mass mixing matrices [4]. CP violating nontrivial complex phases can be allocated in many different ways in 4×4 lepton flavor mixing matrix. Maximal CP violation mainly depends on CP violating phases and mixing angles. In our present article, we have calculated Jarlskog determinant for different sample of θ_x, θ_y and θ_z which are related with CP and T violating relative phases (δ_{ij}).

2 Four Neutrino Mixing Angles and Their Mass Squared Differences

The defect in atmospheric neutrino which depends on zenith-angle, was first observed [5] via transition of $\nu_\mu \rightarrow \nu_\mu$ with $\Delta_{31} = (1-2) \times$

$10^{-3}eV^2, sin^2 2\theta_{23} = 1.0$ mass difference and the mixing respectively. For normal and inverted mass hierarchy, neutrino mass square difference from the three neutrino data analysis of the Super-Kamiokande [6] with 90% CL were observed $1.9 \times 10^{-3}eV^2 \leq \Delta_{31} \leq 2.6 \times 10^{-3}eV^2$ and $1.7 \times 10^{-3}eV^2 \leq \Delta_{31} \leq 2.7 \times 10^{-3}eV^2$, respectively. The another evidence was obtained from the solar neutrino deficit [7], which is consistent with $\nu_\mu \rightarrow \nu_\tau/\nu_e$ transition. The out came from the Sudbury Neutrino Observatory SNO experiments [8] are compatible with the standard solar model [1] and strongly suggest the LMA solution, which is given as $\Delta_{21} = 7 \times 10^{-5}eV^2, sin^2 2\theta_{12} = 0.8$. The others Solar neutrino experiments (Super-K, GALLEX, SAGE, SNO and GNO) show the neutrino oscillations, neutrino oscillation provide the most elegant explanation of all the data [9].

$$\Delta_{solar} = 7^{+5}_{-1.3} \times 10^{-5}eV^2, \tag{1}$$

$$tan^2\theta_{solar} = 0.4^{+0.14}_{-0.1}. \tag{2}$$

Whereas, the analysis of three neutrino global data came from the solar and KamLAND reactor was obtained [10] $\Delta_{21} = 7.5^{+.19}_{-.20} \times 10^{-3}eV^2$ and $tan^2\theta_{12} = 0.452^{+0.035}_{-0.032}$, respectively. Neutrino oscillation is also shown by the atmospheric neutrino experiments (Kamiokande, Super-K) and the best fit to the all data [1] is $\Delta_{atmo} = 2.0^{+1.0}_{-0.92} \times 10^{-3}eV^2$, $sin^2 2\theta_{atmo} = 0.4^{+0.14}_{-0.10}$. The upper bound on the third mixing angle θ_{13} was given by the CHOOZ reactor experiment [11] with the 90 % CL which was found as

$$sin^2\theta_{13} < 0.20 \quad \text{for } |\Delta_{31}| = 2.0 \times 10^{-3}eV^2, \tag{3}$$

$$sin^2\theta_{13} < 0.16 \quad \text{for } |\Delta_{31}| = 2.5 \times 10^{-3}eV^2, \tag{4}$$

$$sin^2\theta_{13} < 0.14 \quad \text{for } |\Delta_{31}| = 3.0 \times 10^{-3}eV^2, \tag{5}$$

The CP phase δ has not been constrained. The third mixing angle θ_{13} and associated mass difference from the two neutrino analysis of the MINOS data was obtained [12] $sin^2\theta_{13} < 0.90$ and $|\Delta_{31}| = (2.43 \pm 0.13) \times 10^{-3}eV^2$, respectively. The future neutrino experiments plan to measure the oscillation parameters precisely. The combined analysis of Day Bay, MINOS and Bugey-3 data [13] has excluded most of the parameter space of the mass squared difference $\Delta_{41} = m_4^2 - m_1^2$ and the mixing angle $sin^2\theta_{14}$ for the sterile neutrino, the latest global analysis of neutrino oscillation data indicates that a small region arround the best-fit value of [6]

$$\Delta_{41} = 1.7eV^2. \quad \text{and} \quad sin^2\theta_{14} = 0.019 \tag{6}$$

3 Jarlskog in Four Flavor Neutrino

The sterile neutrino do not interact via weak interaction. For the flavor mixing of one sterile neutrino (ν_{s}) and three active neutrinos (ν_e, ν_μ, ν_τ), the matrix form of U [14, 15] can be written as

$$U = \begin{pmatrix} U_{s1} & U_{s2} & U_{s3} & U_{s4} \\ U_{e1} & U_{e2} & U_{e3} & U_{e4} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} & U_{\mu 4} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} & U_{\tau 4} \end{pmatrix}, \tag{7}$$

The Four dimension matrix U contains 6 mixing angles and 3 Dirac phase angles and 3 Majorana phase angle. The Jarlskog invariants of CP and T violation is given by:

$$J^{ij}_{\alpha\beta} \approx Im(U_{\alpha i} U_{\beta j} U^*_{\alpha j} U^*_{\beta i}) \tag{8}$$

where,
$\alpha, \beta = s, e, \mu, \tau$ and $i = 1, 2, 3, 4$.

Since U is unitary matrix. So we get,

$$J^{ii}_{\alpha\beta} = J^{ij}_{\alpha\alpha} = J^{ij}_{\beta\beta} = J^{jj}_{\alpha\beta} = 0 \tag{9}$$

and

$$J^{ij}_{\alpha\beta} = -J^{ji}_{\alpha\beta} = J^{ij}_{\beta\alpha} = -J^{ji}_{\beta\alpha} \tag{10}$$

Here we deal with four flavor framework by assuming that the sterile neutrino of eV range and the mixing of this sterile neutrino with three active neutrinos is light. By assuming one sterile neutrinos [23], the PMNS matrix $U_{4\times 4}$ is given by

$$U = R_{34} R_{24} R_{14} R_{23} R_{13} R_{12}, \tag{11}$$

where R_{ij} are rotations matrix in ij space,

$$R_{12} = \begin{pmatrix} c_{12} & s_{12}e^{-i\delta_{12}} & 0 & 0 \\ -s_{12}e^{i\delta_{12}} & c_{12} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and}$$

$$R_{34} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c_{34} & s_{34} \\ 0 & 0 & -s_{34} & c_{34} \end{pmatrix},$$

where $c_{ij} = cos\theta_{ij}, s_{ij} = sin\theta_{ij} e^{i\delta_{ij}}$.

In four flavour the standard form of U is

$$U_{s1} = (c_{14}c_{13}c_{12}),$$

$$U_{s2} = (c_{14}c_{13}s_{12}e^{-i\delta_{12}}),$$

$$U_{s3} = (c_{14}s_{13}e^{-i\delta_{13}}),$$

$$U_{s4} = s_{14}e^{-i\delta_{14}},$$

$$U_{e1} = (-c_{24}c_{23}s_{12}e^{i\delta_{12}} - c_{24}s_{23}e^{-i\delta_{23}}s_{13}e^{i\delta_{13}}c_{12} - s_{24}e^{-i\delta_{24}}s_{14}e^{i\delta_{14}}c_{13}c_{12}),$$

$$U_{e2} = (c_{24}c_{23}c_{12} - c_{24}s_{23}e^{-i\delta_{23}}s_{13}e^{i\delta_{13}}s_{12}e^{-i\delta_{12}} - s_{24}e^{-i\delta_{24}}s_{14}e^{i\delta_{14}}c_{13}s_{12}e^{-i\delta_{12}}),$$

$$U_{e3} = (c_{13}c_{24}s_{23}e^{-i\delta_{23}} - s_{24}e^{-i\delta_{24}}s_{14}e^{i\delta_{14}}s_{13}e^{-i\delta_{13}}),$$

$$U_{e4} = (c_{14}s_{24}e^{-i\delta_{24}}),$$

$$U_{\mu1} = (c_{34}s_{23}e^{i\delta_{23}}s_{12}e^{i\delta_{12}} - c_{34}c_{23}s_{13}e^{i\delta_{13}}c_{12} + s_{34}e^{-i\delta_{34}}s_{24}e^{i\delta_{24}}c_{23}s_{12}e^{i\delta_{12}} + s_{34}e^{-i\delta_{34}}s_{24}e^{i\delta_{24}}s_{23}e^{-i\delta_{23}}s_{13}e^{i\delta_{13}}c_{12} - s_{34}e^{-i\delta_{34}}c_{24}s_{14}e^{i\delta_{14}}c_{13}c_{12}),$$

$$U_{\mu2} = (-c_{34}s_{23}e^{i\delta_{23}}c_{12} - c_{34}c_{23}s_{13}e^{-i\delta_{13}}s_{12}e^{i\delta_{12}} - s_{34}e^{-i\delta_{34}}s_{24}e^{i\delta_{24}}c_{23}c_{12} + s_{34}e^{-i\delta_{34}}s_{24}e^{i\delta_{24}}s_{23}e^{-i\delta_{23}}s_{13}e^{i\delta_{13}}s_{12}e^{-i\delta_{12}} - s_{34}e^{-i\delta_{34}}c_{24}s_{14}e^{i\delta_{14}}c_{13}s_{12}e^{-i\delta_{12}}),$$

$$U_{\mu3} = (c_{34}c_{23}c_{13} - s_{34}e^{-i\delta_{34}}s_{24}e^{i\delta_{24}}s_{23}e^{-i\delta_{23}}c_{13} - s_{34}e^{-i\delta_{34}}c_{24}s_{14}e^{i\delta_{14}}s_{13}e^{-i\delta_{13}})$$

$$U_{\mu4} = (s_{34}e^{-i\delta_{34}}c_{24}c_{14}),$$

$$U_{\tau1} = (s_{34}s_{12}s_{23} + s_{34}e^{i\delta_{34}}c_{23}s_{13}e^{i\delta_{14}}c_{12} + c_{34}s_{24}e^{i\delta_{24}}c_{23}s_{12}e^{i\delta_{12}} + c_{34}s_{24}e^{i\delta_{34}}s_{23}e^{-i\delta_{23}}s_{12}e^{i\delta_{12}}c_{12}$$

$$
\begin{aligned}
&- c_{34}c_{24}s_{14}e^{i\delta_{14}}c_{13}c_{12}),\\
U_{\tau 2} &= (s_{34}e^{i\delta_{34}}c_{12}s_{23}e^{i\delta_{23}} + s_{34}e^{i\delta_{34}}c_{23}s_{13}e^{i\delta_{13}}s_{12}e^{-i\delta_{12}}\\
&- c_{34}s_{24}e^{i\delta_{24}}c_{23}c_{12}\\
&+ c_{34}s_{24}e^{i\delta_{34}}s_{23}e^{-i\delta_{23}}s_{13}e^{i\delta_{13}}s_{12}e^{-i\delta_{12}}\\
&- c_{34}c_{24}s_{14}e^{i\delta_{14}}c_{13}s_{12}e^{-i\delta_{12}}),\\
U_{\tau 3} &= (-s_{34}e^{i\delta_{34}}c_{23}c_{13} - c_{34}s_{24}e^{i\delta_{24}}s_{23}e^{-i\delta_{23}}c_{13}\\
&- c_{34}c_{24}s_{14}e^{i\delta_{14}}s_{13}e^{-i\delta_{13}})\\
U_{\tau 4} &= (c_{34}c_{24}c_{14}),
\end{aligned}
$$

From Equations (9) and (10), we get nine independent $J^{ij}_{\alpha\beta}$, the magnitudes of $J^{ij}_{\alpha\beta}$ depends on mixing angles and six CP-violating phases. The explicit form of nine jarlskog $J^{ij}_{\alpha\beta}$ are

$$
\begin{aligned}
J^{13}_{se} &= c_{12}c_{13}c^2_{14}c_{23}c_{24}s_{12}s^2_{13}s_{14}s_{24}sin\theta_y\\
&- c_{12}c^2_{13}c^2_{14}c_{23}c^2_{24}s_{12}s_{13}s_{14}s_{23}sin\theta_z\\
&+ c^2_{12}c_{13}c^2_{14}c_{24}s_{24}s_{13}s_{14}s_{23}sin\left(\theta_y - \theta_z\right) \qquad (12)
\end{aligned}
$$

$$
\begin{aligned}
J^{24}_{se} &= c_{12}c_{13}c^2_{14}c_{23}c_{24}s_{12}s_{14}s_{24}sin\theta_y\\
&- c_{13}c^2_{14}c_{24}s^2_{12}s_{13}s_{14}s_{23}s_{24}sin\left(\theta_y - \theta_z\right) \qquad (13)
\end{aligned}
$$

$$
J^{34}_{se} = c_{13}c^2_{14}c_{24}s_{13}s_{14}s_{23}s_{24}sin\left(\theta_y - \theta_z\right) \qquad (14)
$$

$$
\begin{aligned}
J^{13}_{\tau s} &= c^2_{12}c_{13}c^2_{14}c_{23}c_{24}c_{34}s_{13}s_{14}s_{34}sin\theta_x\\
&+ c_{12}c_{13}c^2_{14}c_{23}c_{24}c^2_{34}s_{12}s^2_{13}s_{14}s_{24}sin\theta_y\\
&+ \left(c^2_{34}s^2_{24} - s^2_{34}\right) c_{12}c^2_{13}c^2_{14}c_{23}s_{12}s_{13}s_{23}sin\theta_z\\
&- c_{12}c^2_{13}c^2_{14}c^2_{23}c_{34}s_{12}s_{13}s_{24}s_{34}sin\left(\theta_x - \theta_y\right)\\
&- c_{12}c_{13}c^2_{14}c_{24}c_{34}s_{12}s^2_{13}s_{14}s_{23}s_{34}sin\left(\theta_x + \theta_z\right)\\
&+ c^2_{12}c_{13}c^2_{14}c^2_{34}c_{24}s_{13}s_{14}s_{23}s_{24}sin\left(\theta_y - \theta_z\right)\\
&- c_{12}c^2_{13}c^2_{14}c_{34}s_{12}s_{13}s^2_{23}s_{24}s_{34}sin\left(\theta_x - \theta_y + \theta_z\right) \qquad (15)
\end{aligned}
$$

$$\begin{aligned} J_{\tau s}^{14} = &-c_{12}^2 c_{13} c_{14}^2 c_{23} c_{24} c_{34} s_{13} s_{14} s_{34} sin\theta_x \\ &- c_{12} c_{13} c_{14}^2 c_{23} c_{24} c_{34}^2 s_{12} s_{13} s_{24} sin\theta_y \\ &+ c_{12} c_{13} c_{14}^2 c_{24} c_{34} s_{12} s_{14} s_{23} s_{34} sin\left(\theta_x + \theta_z\right) \\ &- c_{12}^2 c_{13} c_{14}^2 c_{24} c_{34}^2 s_{12} s_{13} s_{23} s_{24} sin\left(\theta_y - \theta_z\right) \end{aligned} \tag{16}$$

$$\begin{aligned} J_{\tau s}^{34} = &\, c_{13} c_{14}^2 c_{23} c_{24} c_{34} s_{13} s_{14} s_{34} sin\theta_x \\ &+ c_{13} c_{14}^2 c_{24} c_{34}^2 s_{13} s_{14} s_{23} s_{24} sin\left(\theta_y - \theta_z\right) \end{aligned} \tag{17}$$

and

$$\begin{aligned} J_{e\mu}^{23} = &-\left(c_{12}^2 c_{23}^2 s_{24}^2 - c_{12}^2 c_{24}^2 s_{23}^2 - s_{12}^2 s_{14}^2 s_{24}^2 + s_{12}^2 s_{23}^2\right) \\ &\times c_{13} c_{23} c_{24} c_{34} s_{13} s_{14} s_{34} sin\theta_x \\ &+ \left(c_{13}^2 c_{34}^2 s_{23}^2 - c_{23}^2 c_{34}^2 s_{13}^2 + s_{13}^2 s_{14}^2 s_{34}^2 - s_{34}^2 s_{23}^2\right) \\ &\times c_{12} c_{13} c_{23} c_{24} s_{12} s_{14} s_{24} sin\theta_y \\ &+ \left(c_{24}^2 c_{34}^2 - s_{14}^2 c_{24}^2 s_{34}^2 - c_{34}^2 s_{14}^2 s_{24}^2 + s_{14}^2 s_{24}^2 s_{34}^2\right) \\ &\times c_{12} c_{13}^2 c_{23} c_{34} s_{12} s_{13} s_{23} sin\theta_z \\ &+ \left(c_{13}^2 s_{24}^2 - c_{24}^2 s_{13}^2\right) \\ &\times c_{12} c_{23}^2 c_{34} s_{12} s_{13} s_{14}^2 s_{24} s_{34} sin\left(\theta_x - \theta_y\right) \\ &+ \left(c_{23}^2 - c_{13}^2 s_{24}^2 - c_{24}^2 s_{13}^2 + s_{13}^2 s_{14}^2 s_{24}^2\right) \\ &\times c_{12} c_{13} c_{24} c_{34} s_{12} s_{14} s_{23} s_{34} sin\left(\theta_x + \theta_z\right) \\ &- \left(c_{12}^2 c_{23}^2 c_{34}^2 - c_{12}^2 c_{23}^2 s_{34}^2 - c_{23}^2 c_{34}^2 s_{12}^2 + s_{12}^2 s_{14}^2 s_{34}^2 - s_{12}^2 s_{23}^2 s_{34}^2\right) \\ &\times c_{13} c_{24} s_{13} s_{14} s_{23} s_{24} sin\left(\theta_y - \theta_z\right) \\ &+ \left(c_{12}^2 c_{13}^2 c_{24}^2 - c_{12}^2 c_{24}^2 s_{13}^2 s_{14}^2 - c_{13}^2 c_{24}^2 s_{12}^2 s_{14}^2 + c_{24}^2 s_{12}^2 s_{13}^2 s_{14}^2\right) \\ &\times c_{23} c_{34} s_{23} s_{24} s_{34} sin\left(\theta_x - \theta_y + \theta_z\right) \\ &- \left(c_{13}^2 c_{24}^2 s_{23}^2 - c_{13}^2 s_{23}^2 s_{24}^2 s_{14}^2 - c_{24}^2 s_{23}^2 s_{13}^2 s_{14}^2\right) \\ &\times c_{12} c_{34} s_{12} s_{13} s_{24} s_{34} sin\left(\theta_x - \theta_y + 2\theta_z\right) \\ &- c_{12} c_{13}^2 c_{24}^2 c_{34} s_{12} s_{13} s_{14}^2 s_{24} s_{34} sin\left(\theta_x + \theta_y\right) \end{aligned}$$

$$
\begin{aligned}
&+ c_{12}c_{13}c_{23}^2c_{24}c_{34}s_{12}s_{13}^2s_{14}s_{23}s_{34}sin\left(\theta_x - \theta_z\right)\\
&+ \left(c_{34}^2 - s_{34}^2\right)c_{12}c_{13}c_{23}c_{24}s_{12}s_{13}^2s_{14}s_{23}^2s_{24}sin\left(\theta_y - 2\theta_z\right)\\
&- \left(c_{12}^2 - s_{12}^2\right)c_{13}c_{23}c_{24}c_{34}s_{13}s_{24}^2s_{14}s_{23}^2s_{34}sin\left(\theta_x - 2\theta_y + 2\theta_z\right)\\
&- c_{12}c_{13}c_{23}^2c_{24}c_{34}s_{12}s_{24}^2s_{14}s_{23}s_{34}sin\left(\theta_x - 2\theta_y + \theta_z\right)\\
&+ c_{12}c_{13}c_{24}c_{34}s_{12}s_{13}^2s_{14}s_{23}^2s_{24}^2s_{34}sin\left(\theta_x - 2\theta_y + 3\theta_z\right) \qquad (18)
\end{aligned}
$$

$$
\begin{aligned}
J_{e\mu}^{24} = &\; c_{13}c_{14}^2c_{23}c_{24}c_{34}s_{12}^2s_{13}s_{14}s_{24}^2s_{34}sin\theta_x\\
&+ c_{12}c_{13}c_{14}^2c_{23}c_{24}s_{34}^2s_{12}s_{14}s_{24}sin\theta_y\\
&- c_{12}c_{14}^2c_{23}^2c_{24}^2c_{34}s_{12}s_{13}s_{24}s_{34}sin\left(\theta_x - \theta_y\right)\\
&+ c_{12}c_{13}c_{14}^2c_{24}c_{34}s_{12}s_{14}s_{23}s_{24}^2s_{34}sin\left(\theta_x + \theta_y\right)\\
&- c_{13}c_{14}^2c_{24}s_{12}^2s_{13}s_{14}s_{23}s_{24}s_{34}^2sin\left(\theta_y - \theta_z\right)\\
&+ c_{12}c_{13}^2c_{14}^2c_{34}s_{12}s_{13}s_{23}^2s_{24}s_{34}sin\left(\theta_x - \theta_y + 2\theta_z\right)\\
&- \left(c_{12}^2 - s_{12}^2s_{13}^2\right)c_{14}^2c_{23}c_{24}^2c_{34}s_{23}s_{24}s_{34}sin\left(\theta_x - \theta_y + \theta_z\right) \qquad (19)
\end{aligned}
$$

$$
\begin{aligned}
J_{e\mu}^{34} = &-c_{13}c_{14}^2c_{23}c_{24}c_{34}s_{13}s_{14}s_{24}^2s_{34}sin\theta_x\\
&+ c_{13}c_{14}^2c_{24}s_{13}s_{14}s_{23}s_{24}s_{34}^2sin\left(\theta_y - \theta_z\right)\\
&+ c_{13}^2c_{14}^2c_{23}c_{24}^2c_{34}s_{23}s_{24}s_{34}sin\left(\theta_x - \theta_y + \theta_z\right) \qquad (20)
\end{aligned}
$$

where

$$
\begin{aligned}
\theta_x &= \delta_{14} - \delta_{13} - \delta_{34}\\
\theta_y &= \delta_{14} - \delta_{12} - \delta_{24} \qquad (21)\\
\theta_z &= \delta_{13} - \delta_{12} - \delta_{23}
\end{aligned}
$$

We assume the values of 6 CP-violating phases δ_{ij} are between 0 and 2π. Equations (12–20) are important to study the role of violation in T and CP within the framework of four-neutrino mixing. B. S. Koranga et al. [16, 18, 19, 22, 24] did some analysis on T and CP violation within three flavor framework for different parameterization and above GUT scale.

4 Results and Discussion

In numerical calculation, The active sterile neutrino mixing angle are θ_{14}, θ_{24} and θ_{34}. In this calculation, we consider following value for sterile neutrino mixing angles [17], $\theta_{14} = 3.6°$, $\theta_{24} = 4°$, $\theta_{34} = 18.5°$. We consider mixing angles $\theta_{13} = 10°$, $\theta_{23} = 45°$, $\theta_{12} = 34°$, $\theta_{34} = 18.5°$, $\theta_{24} = 4°$, $\theta_{14} = 3.6°$ [17]. The calculated values of Jarlskog determinant for different sample of θ_x, θ_y and θ_z are presented in Tables 1 and 2. These three θ_x, θ_y and θ_z angles are directly related with six Dirac CP phases which is associated with their mixing angles. On varying these three angles we obtained different values of nine Jarlskog determinant from which we took the maximum and the minimum value of Jarlskog determinant. The upper bound value of $J^{23}_{e\mu} = 0.04510635487051475$ for $\theta_x = 31°, \theta_y = 28°$ and $\theta_z = 90°$ and lower bound of $J^{13}_{se} = -0.039055825709320924$ for $\theta_x = 0°, \theta_y = 0°$ and $\theta_z = 90°$ are presented in Tables 1 and 2 respectively.

Table 1 Nine maximum Jarlskog Determinant values for various value of mixing angle. Current value of mixing angles $\theta_{23} = 45°$, $\theta_{10} = 10°$, $\theta_{12} = 34°$, $\theta_{34} = 18.5°$, $\theta_{14} = 3.6°$, $\theta_{24} = 4°$

$J^{ij}_{\alpha\beta}$	Maximum	θ_x	θ_y	θ_z
$J^{13}_{\tau s}$	0.036685890619149106	98°	170°	91°
$J^{14}_{\tau s}$	0.014752208075303027	0°	90°	90°
$J^{34}_{\tau s}$	0.002743458805405071	90°	90°	0°
J^{13}_{se}	0.00040407451857360076	0°	90°	0°
J^{24}_{se}	0.0015695627093747612	0°	90°	180°
J^{34}_{se}	0.0005262721896176136	0°	90°	0°
$J^{23}_{e\mu}$	0.04510635487051475	31°	28°	90°
$J^{24}_{e\mu}$	0.007314874300972153	0°	150°	74°
$J^{34}_{e\mu}$	0.010142178442526652	180°	90°	0°

Table 2 Nine minimum Jarlskog Determinant values for various value of mixing angle. Current value of mixing angles $\theta_{23} = 45°$, $\theta_{10} = 10°$, $\theta_{12} = 34°$, $\theta_{34} = 18.5°$, $\theta_{14} = 3.6°$, $\theta_{24} = 4°$

$J^{ij}_{\alpha\beta}$	Minimun	θ_x	θ_y	θ_z
$J^{13}_{\tau s}$	−0.0019163137495424884	180°	90°	180°
$J^{14}_{\tau s}$	−0.018052733828834214	128°	139°	38°
$J^{34}_{\tau s}$	−0.0004732859240982911\5	0°	0°	90°
J^{13}_{se}	−0.039055825709320924	0°	0°	90°
J^{24}_{se}	−0.00016456357872771316	0°	180°	90°
J^{34}_{se}	−0.0005262721896176136	0°	0°	90°
$J^{23}_{e\mu}$	−0.009549376384925656	90°	0°	180°
$J^{24}_{e\mu}$	−0.007283419390778245	180°	180°	102°
$J^{34}_{e\mu}$	−0.010143317279290433	168°	0°	102°

5 Conclusions

Jarlskog invariance plays a crucial role in developing a better understanding of flavors of neutrino. It has potency to tune the magnitude of CP and T violation in both quark and lepton sectors. In present article, we have calculated the upper and lower bound of Jarlskog Determinant in a four flavor framework. The upper bound and lower bound values of Jarlskog determinant is found out to be $J_{e\mu}^{23} = 0.04510635487051475$ for $\theta_x = 31°, \theta_y = 28°$ and $\theta_z = 90°$ and $J_{se}^{13} = -0.039055825709320924$ for $\theta_x = 0°, \theta_y = 0°$ and $\theta_z = 90°$ respectively. Since, these Jarlskog determinant appear in the imaginary part of the expression of oscillation probability amplitude [15, 25], hence it does not contribute to oscillation probability. Whereas it plays a crucial role when we deal with the leptonic sector CP and T violation. The contribution of Jarlskog determinant $J_{e\mu}^{23}$ and $J_{e\mu}^{13}$ in CP violation, found to be more significant among the others nine Jarlskog determinant.

Acknowledgement

One of the authors BSK thanks Ratindra Gautam for the preperation of manuscript.

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